Quantitative Policy Analysis for Sustainable Development in Water-Stressed Developing Countries

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1. Introduction

The water crisis is a highly complex problem. It is characterised by its multi-dimensionality encompassing a wide variety of problems such as water shortage, water pollution, public health and food security all of which have social, ecological, cultural, and economic dimensions (Saleth 2002). As the success of any model depends upon a balance between realism and analytical tractability (Intriligator 1983), it is no wonder that building a useful model for this highly complex problem is a challenging task. On the other hand, modelling is a potentially powerful and promising approach for this issue exactly because of this complexity.

The principal objective of my PhD research is to conduct quantitative policy analysis with respect to water problems in water-stressed developing countries in order to have policy prescription for sustainable development.

In this paper a stylised Ramsey-Cass-Koopmans (RCK) growth model is developed as the analytic model of the quantitative policy analysis. On the surface, the specification of this analytic model does not seem to be adequate for capturing some key stylised facts of water-stressed underdeveloped countries, such as vulnerability of rain-fed agriculture, high unemployment rate, and so on. The fact is that the analytic model stands as the simplified version of more general model reflecting these stylised facts. The major roles of the analytic model are (i) to provide a model platform on which more general model is constructed, and (ii) to clarify policy implications of these stylised facts on sustainable development by investigating the case without them.

The structure of this paper is as follows. Section 2 explains general features of the analytic model, in particular the specification of two-stage dynamic optimisation without perfect foresight assumption. Section 3 shows the results of the first-stage optimisation in which households and private firms optimise their objective functions. Section 4 shows the results of the second-stage optimisation by the government. Section 5 presents the qualitative analysis of the predicted optimal time paths. Section 6 provides conclusion of this paper.

2. Outline of the Analytic Model

The analytic RCK growth model is designed so as to be compatible with the more generalised multi-sector growth model incorporating the following stylised facts of the water scarce developing economies.

- Irrigation shares vast majority of total water use, often reaching 80 to 90%.
- Production risks of rain-fed agriculture are one of the main causes of rural poverty and consequently of rural-urban migration.
- An urban unemployment rate remains high with considerable rural-urban migration in spite of priority public investments in urban modern sectors.
- A lack of safe water access, which is rather common in the rural areas or in the urban squatter areas, severely undermines the social welfare through various pathways, via direct and indirect health risks and higher medical and water expenditure, or via depriving educational opportunities from children.

The compatibility between models means that the generalised version can be obtained by relaxing some assumptions of the analytic model. More specifically, both models have essentially the same control variables, i.e. consumption levels of the market good and domestic water for the household, factor inputs including water for the private firms, and public investment as well as the water price for the public water producer. The stylised facts abovementioned are abstracted from the analytic model in order to have benchmark results based on which we can examine the implication of each stylised fact.

The analytic model assumes a closed economy consisting of numerous identical households and identical competitive firms of which output is the numeraire of the economy. There exist population growth (as the growth of household size) at a constant rate ν and capital depreciation at a constant rate δ. Further, it is assumed that a budget neutral government provides water to households and private producers and collects a volumetric water charge.

The social optimisation process consists of two stages. At the first stage, the households maximise their utility by choosing consumption levels and the private firms maximise their profits by choosing the amounts of factor inputs taking the rate of water charge as given. At the second stage, the government maximises the social welfare by choosing the rate of water charge and by investing the collected water charge in public capital that is the sole factor input of water supply service. In addition to the rationales mentioned in the previous chapter, this specification allows the government intervention to be dynamically efficient.

Another novelty of the present model is the households' expectation process without the perfect foresight assumption. The conventional RCK growth models assume that the households determine the optimal consumption trajectory by deriving optimal conditions of instantaneous rates of change of consumption, with determining the optimal initial consumption. The latter is determined based on the consumption function derived from the intertemporal budget constraint, with an assumption that the households can precisely predict the trajectories of the wage rate, the water price, and the interest rate. The analytic model in this thesis attempts to liberate the households from this perfect foresight assumption. In this model the households make their decision of consumption level based on their expectation of the future trajectories of those exogenous variables, but they do not believe their expectation precisely forecast them. Instead, the households continuously modify their expectation based on the realised levels of these exogenous variables. The realised consumption trajectory satisfies the second-best optimality. At the optimal steady state the second-best outcomes coincide the first-best outcomes.

3. First-stage Optimisation

The first-stage optimisation consists of households' utility maximisation and firms' profit maximisation taking the government policy in terms of water price as exogenously given.

3.1 Household's problem

(1) Problem formulation

It is assumed that households hold assets as equity shares of the private capital stock. Population is defined as the labour force population and each person supplies one unit of labour services per unit of time. [1] Households earn wage and capital income, purchase publicly supplied water and manufactured goods for consumption, and invest in private capital stock. As a result the per capita budget constraint of the representative household becomes $w + rm = c_M + pq_H + I$, where w is the wage rate, r is the real rate of return to equity shares, m is the household assets, c_M is per capita consumption of the manufactured good, q_H is per capita domestic water consumption, p is the rate of water charge (water price), and I is the household investment in equity shares. [2]

The equation of motion of per capita equity shares owned by a household is $\dot{m} = I - vm$. [3] The latter term corresponds to "dilution" due to growth of household size (Aghion and Howitt 1998; p.14).

These two equations merge into

$$\dot{m} = w + (r - v)m - c_M - pq_H \,. \tag{1}$$

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[1] It means that we assume the same proportion between consumption of labour force age person and that of his/her dependents such as young children and elderly people throughout time horizon. In other words, a person in our model consists of one labour force age person and his/her dependents. In empirical analysis this assumption is important.

[2] All variables are time variant, i.e. $w(t)$, $r(t)$, etc., but time is omitted for notational simplicity.

[3] Superimposed dot means time derivatives. Note that the equity shares do not depreciate though the corresponding private capital does.

It is assumed that households' utility at time t is determined by the discounted sum of felicities for certain length of period T.[4] The felicity function is assumed to be CIES (constant intertemporal elasticity of substitution) type.[5]

$$U(t) = \int_t^{t+T} e^{-(\rho-\nu)s} u(c(s))ds\,,\ u(c(t)) \equiv \frac{\{c(t)\}^{1-\sigma}}{1-\sigma}\,,$$

where $c(t)$: the consumption level of flow of satisfaction produced by the household itself at time t, T: length of planning period, ρ: the rate of pure time preference, and σ: the elasticity of marginal felicity with respect to consumption.[6]

The underlying assumption is that an immortal household consists of continuously distributed age groups and that the terminal time of planning horizon continuously shifts forward. This is a straightforward extension of the utility maximisation problem of an individual. We have little idea about the length T that might be formed by economic and social circumstances as well as education in the real world.[7] For the sake of analytical simplicity T is assumed to be infinite, which may reduce the accuracy of households expectation but not significantly with time discounting. When we employ a finite T, say 20 years, we have to specify the terminal condition for the household assets as well. Both the choice of T and that of the terminal condition are of quite arbitrary nature.

It is assumed that households produce a flow of satisfaction by consuming the manufacturing good and water.

$$c(c_M, q_H) \equiv c_M{}^{\varphi} q_H{}^{1-\varphi}\,,\ 0 < \varphi < 1, \tag{2}$$

where φ is a weight of manufacturing good in satisfaction production.

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4 The common notion of "instantaneous utility" is deliberately avoided. In my thesis felicity is analogous to the notion of ophelimity in the works of Pareto. Pareto defined ophelimity as satisfaction derived from economic activities which is merely an ingredient of utility (happiness). For further discussion of ophelimity/utility distinction in Pareto, see Tarascio (1969).

5 CIES functions are functionally identical with CRRA (constant relative risk aversion) functions. This specification is advantageous since later we will introduce risks in our generalised models.

6 In this thesis ρ - ν > 0 and σ > 1 are assumed. Arrow et al. (1996) report that majority of studies use values in the range of 1 to 2 for σ.

7 Perrings argues that poverty may drive up poor farmers extremely myopic such that "all that matters is consumption today" (Perrings 1989; p.20). Becker and Mulligan (1997)'s argument on time preference formation is applicable to this time horizon determination as well.

Each household maximises its utility subject to budget constraint taking the water price as given. Hence the representative household's optimisation problem at time *t* is

$$\underset{c_M, q_H}{Max}\ U(t) \equiv \int_t^{\infty} e^{-(\rho-\nu)s} \frac{c(c_M(s), q_H(s))^{(1-\sigma)}}{1-\sigma} ds, \quad \text{subject to}$$

$$\dot{m} = w + (r-\nu)m - c_M - pq_H, \text{ and}$$

the initial assets $m(t)$ is historically determined at time *t*.

(2) Optimal growth rate of consumption

The current value Hamiltonian of this problem is

$$\tilde{H} = \frac{c(c_M, q_H)^{(1-\sigma)}}{1-\sigma} + \lambda\{w + (r-\nu)m - c_M - pq_H\},$$

where λ is the Lagrange multiplier associated with household assets *m*.

Assuming an interior solution, the necessary and sufficient conditions are as follows.[8]

$$\frac{\partial \tilde{H}}{\partial c_M} = 0 \;\Rightarrow\; \lambda = \varphi \frac{c^{1-\sigma}}{c_M} \tag{3a}$$

$$\frac{\partial \tilde{H}}{\partial q_H} = 0 \;\Rightarrow\; p\lambda = (1-\varphi)\frac{c^{1-\sigma}}{q_H} \tag{3b}$$

$$\dot{\lambda} - (\rho-\nu)\lambda = -\frac{\partial \tilde{H}}{\partial m} \;\Rightarrow\; \frac{\dot{\lambda}}{\lambda} = -(r-\rho) \tag{3c}$$

In addition, the transversality condition is $\lim_{s\to\infty}\left[e^{-(\rho-\nu)(s-t)}\hat{\lambda}(s)\cdot\hat{m}(s)\right] = 0$.[9]

From (2), (3a) and (3b) we derive

$$q_H = \left(\frac{\varphi}{1-\varphi}\right)^{-\varphi} p^{-\varphi} c, \text{ and } c_M = \left(\frac{\varphi}{1-\varphi}\right)^{1-\varphi} p^{1-\varphi} c. \tag{4}$$

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8 Since each of the objective function and the constraint is a concave function associated with a negative semidefinite Hessian matrix, the Mangasarian Sufficiency Theorem (Mangasarian 1966) can be applied.

9 " ^ " denotes the first stage solution.

By putting (4) into (3b) we have $\lambda = \varphi^{\varphi}(1-\varphi)^{1-\varphi} p^{-(1-\varphi)} c^{-\sigma}$. By taking time derivative of the both sides with logarithmic transformation, we obtain $\frac{\dot{\lambda}}{\lambda} = -(1-\varphi)\frac{\dot{p}}{p} - \sigma\frac{\dot{c}}{c}$. From this equation and Eq. (3c) we derive the optimal growth rate of consumption as

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left\{(r-\rho) - (1-\varphi)\frac{\dot{p}}{p}\right\}. \tag{5}$$

The difference between this result and that of the standard RCK model lies in the far right term. There exists some negative effect of the water price rise on the consumption growth. The larger the weight of water consumption in satisfaction production $(1-\varphi)$ is, the severer this effect is. It might be plausible that the developing economies would be more sensitive to this negative impact of water price rise on consumption growth, due to higher share of water expenditure among the total household expenditure.

(3) Optimal consumption level

In order to determine the optimal consumption level, we need to construct the consumption function based on the intertemporal budget constraint. With the optimality conditions (4), the equation of motion of the household's assets becomes $\dot{m} = w + (r - \nu)m - b_1 p^{1-\varphi} c$, where $b_1 \equiv \varphi^{-\varphi}(1-\varphi)^{-(1-\varphi)} > 0$.

The solution of this differential equation for the period between t and $t + T$ is

$$m(t+T)e^{-\int_t^{t+T}\{r(\tau)-\nu\}d\tau} = m(t) + \int_t^{t+T}\left[w(s) - b_1\{p(s)\}^{1-\varphi} c(s)\right]e^{-\int_t^{s}\{r(\tau)-\nu\}d\tau} ds.$$

When we take the limit as T approaches infinity, the left hand side becomes zero from the transversality condition. Thus, the intertemporal budget constraint becomes

$$b_1\int_t^{\infty}\{p(s)\}^{1-\varphi} c(s)\, e^{-\int_t^{s}\{r(\tau)-\nu\}d\tau} ds = m(t) + \int_t^{\infty} w(s)\, e^{-\int_t^{s}\{r(\tau)-\nu\}d\tau} ds.$$

The left hand side is the present value of the household's total spending, while the right hand side is wealth defined as the sum of the disposable assets and the present value of wage income. The following expression of consumption is obtained by solving Eq. (5).

$$c(s)=c(t)\left\{\frac{p(t)}{p(s)}\right\}^{b_2}\exp\left[\frac{1}{\sigma}\int_t^s\{r(\tau)-\rho\}d\tau\right], \text{ where } b_2\equiv\frac{1-\varphi}{\sigma}>0.$$

From these two equations we derive the following consumption function of the "clairvoyant" household who can predict the future trajectories of w, r, and p perfectly.

$$c(t)=\eta(t)\left[m(t)+\int_t^\infty w(s)e^{-\int_t^s\{r(\tau)-\nu\}d\tau}ds\right], \text{ where}$$

$$\eta(t)\equiv\left[b_1\{p(t)\}^{b_2}\int_t^\infty\{p(s)\}^{b_3}e^{\int_t^s\{b_4-b_5r(\tau)\}d\tau}ds\right]^{-1},$$

in which $b_3\equiv\frac{(\sigma-1)(1-\varphi)}{\sigma}>0$ and $b_4\equiv\nu-\frac{\rho}{\sigma}$ and $b_5\equiv\frac{\sigma-1}{\sigma}>0$.

The term $\eta(t)$ is the propensity to consume out of wealth at period t. It is noted that the clairvoyant households need to use the consumption function only once when they choose the initial consumption at t = 0, then they just need to change the level of consumption based on the optimal consumption growth rate expressed as Eq. (5) in order to achieve the first-best optimality.

Now let's relax the perfect foresight assumption. Instead, it is assumed that the households' expectation about the trajectories of exogenous variables is that they are constant at their current values.[10] With this assumption the following proposition is derived.

Proposition 1: Optimal consumption level

If $r(t)>\nu$ is satisfied, the optimal consumption is given as

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[10] It is also possible to incorporate past information in the expectation formation process. For instance, the expected trajectory might grow at constant rate estimated based on the past growth rates. I feel, however, that this kind of sophistication is of ad-hoc nature anyway and its rewards might not be enough to compensate its costs.

$$\hat{c}(t)=\frac{\{b_5 r(t)-b_4\}}{b_1\{p(t)\}^{1-\varphi}}\left\{m(t)+\frac{w(t)}{r(t)-\nu}\right\}.$$

Otherwise $\hat{c}(t)$ diverges towards either negative or positive infinity.

Proof: See Appendix A1.

In the following analysis, $r > \nu$ is always assumed. Now $\hat{c}(t)$ is determined solely by the contemporaneous values of exogenous variables. Though the trajectory of $\hat{c}(t)$ is different from that of the clairvoyant households' optimal unless the economy is at the steady-state, it is the optimal trajectory of the household consumption given the expectation formation process.

3.2 Firm's problem

We assume that the representative firm's production technology is described as the following Cobb-Douglas production function with constant return to scale.

$$Y = F(K, Q_M, L) = K^{\beta_K} Q_M{}^{\beta_Q} L^{\beta_L}$$

where Y: output, K: private capital stock, and Q_M: water input, L: labour input, and β_K, β_Q, and β_L: factor shares of private capital, water and labour with $\beta_K, \beta_Q, \beta_L \in (0,1)$ and $\beta_K + \beta_Q + \beta_L = 1$.

The constant return to scale assumption enables us to express the above production technology in the following intensive form.

$$y = f(k, q_M) = k^{\beta_K} q_M{}^{\beta_Q},$$

where y: per worker output, k: per worker private capital stock, and q_M: per worker water input.

The firm's per worker profit is

$$\pi = y - (r+\delta)k - pq_M - w, \tag{6}$$

where π is per worker profit and p is the water price.[11]

The representative firm maximises per worker profit by setting the partial derivatives of π with respect to k and q_M at zero, taking r and p as exogenously given.

$$\frac{\partial \pi}{\partial k} = 0 \;\Rightarrow\; \beta_K y = (r + \delta)k \text{ , and } \frac{\partial \pi}{\partial q_M} = 0 \;\Rightarrow\; \beta_Q y = p q_M \qquad (7)$$

From the per worker production function and the above optimal conditions we can express y and q_M as a function of k and p.

$$y = \beta_Q{}^{b_6} p^{-b_6} k^{b_7} \text{ , and } q_M = \beta_Q{}^{\frac{1}{1-\beta_Q}} p^{\frac{1}{\beta_Q - 1}} k^{b_7} \text{ ,} \qquad (8)$$

where $b_6 \equiv \dfrac{\beta_Q}{1-\beta_Q} > 0$ and $b_7 \equiv \dfrac{\beta_K}{1-\beta_Q} > 0$.

3.3 Market equilibrium

The equilibrium of the labour, the capital and the good markets is achieved by a set of prices r^* and w^* such that these markets are clear.[12]

The equilibrium wage rate w^* clears the labour market such that per worker profit equals zero and total (labour force) population equals number of total workers. Hence, at the equilibrium per capita values and per worker values coincide. By putting Eq. (7) into the per worker profit function (6) with applying the zero optimal profit condition, we obtain the equilibrium wage rate as $w^* = \beta_L y = \beta_L \beta_Q{}^{b_6} p^{-b_6} k^{b7}$.

The equilibrium rate of return to private capital r^* clears the private capital market such that the supply and the demand of aggregate capital coincide. If the labour market clears at the same time, r^* is determined such that per capita household assets become equal to per worker private capital stock, i.e. $\hat{k} = \hat{m}$.

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[11] See footnote 4. Too compensate the depreciation of capital the rental price of capital must be $r + \delta$ (see, e.g. Barro and Sala-i-Martin 1995: p.69).

[12] Since the market good is numeraire its equilibrium price is always unity.

Since the private firms can realise their optimal output with r^*, we can express r^* as a function of $\hat{k}$ and p from Eq. (7) as

$$r^* = \beta_K \beta_Q{}^{b_6} p^{-b_6} \hat{k}^{-b_8} - \delta \text{ , where } b_8 \equiv \frac{\beta_L}{1-\beta_Q} > 0 .$$

3.4 First-stage solution

By putting w^* and r^* into the optimal consumption level in the proposition 1 and substitute $\hat{m}$ with $\hat{k}$, the optimal consumption can be expressed as follows.

$$\hat{c} = \hat{k}\frac{\left(b_{10} p^{-b_6} \hat{k}^{-b_8} - b_9\right)}{b_1 p^{1-\varphi}}\left(1 + \frac{\beta_L}{\beta_K - b_{11} p^{b_6} \hat{k}^{b_8}}\right), \tag{9}$$

where $b_9 \equiv \delta + \nu - \dfrac{\delta + \rho}{\sigma}$, $b_{10} \equiv \dfrac{(\sigma - 1)\beta_K \beta_Q{}^{\frac{\beta_Q}{1-\beta_Q}}}{\sigma} > 0$, and

$b_{11} \equiv (\delta + \nu)\beta_Q{}^{\frac{\beta_Q}{\beta_Q - 1}} > 0$.

The following equation of motion of the private capital is derived from equations (1), (4) and (7).

$$\dot{\hat{k}} = \left(1 - \beta_Q\right)\hat{y} - (\delta + \nu)\hat{k} - b_1 p^{1-\varphi}\hat{c} \text{ , } \hat{k}(0) = k_0 \text{ (} = m_0 \text{, given).}$$

From the above equation of motion with the equations (8) and (9) the optimal growth rate of private capital becomes as follows.

$$\frac{\dot{\hat{k}}}{\hat{k}} = \frac{b_{12}\left(p^{b_6}\hat{k}^{b_8}\right)^2 - b_{13} p^{b_6}\hat{k}^{b_8} + b_{14}}{\sigma\left(\beta_K - b_{11} p^{b_6}\hat{k}^{b_8}\right)p^{b_6}\hat{k}^{b_8}} \equiv \phi^k\left(\hat{k}, p,\right), \ \hat{k}(0) = k_{0,} \tag{10}$$

where $b_{12} \equiv (\delta + \nu)(\delta + \rho)\beta_Q{}^{\frac{\beta_Q}{\beta_Q - 1}} > 0$,

$b_{13} \equiv (\delta + \nu)\beta_K + (\delta + \rho)\left(1 - \beta_Q\right) > 0$, and $b_{14} \equiv \beta_K\left(1 - \beta_Q\right)\beta_Q{}^{\frac{\beta_Q}{1-\beta_Q}} > 0$.

Taking the trajectory of the water price as exogenously given, the private firms determine the optimal stock level of private capital based on this equation. Let's introduce $\xi \equiv p^{b_6}\hat{k}^{b_8}$ and rewrite Eq. (10) as $\phi^k(\xi) \equiv \dfrac{b_{12}\xi^2 - b_{13}\xi + b_{14}}{\sigma(\beta_K - b_{11}\xi)\xi}$.

The condition $r > \nu$ and $r^* = \beta_K \beta_Q^{\frac{\beta_Q}{1-\beta_Q}}\xi^{-1} - \delta$ determine the domain of $\phi^k(\xi)$ as

$0 < \xi < \dfrac{\beta_K}{\delta + \nu}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}} \equiv \xi_{\max}$.[13]

Within this domain, the following lemma holds.

Lemma 1:

The sign of growth rate of capital is determined by the following rule.

$$\phi^k(\xi) \begin{matrix} > \\ = \\ < \end{matrix} 0, \text{ if and only if } \xi \begin{matrix} < \\ = \\ > \end{matrix} \bar{\xi} \equiv \frac{\beta_K}{\delta + \rho}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}.$$

Moreover, $\lim_{\xi \to 0+} \phi^k(\xi) = \infty$, and $\lim_{\xi \to \xi_{\max}-} \phi^k(\xi) = -\infty$.

Proof: See Appendix A2.

Now we examine the effects of the water price on the trajectory of $\hat{k}$. Remind that $p(t)$ does not affect $\hat{k}(t)$ although $p(t)$ affects the growth rate of $\hat{k}(t)$. It means that we can freely change $\xi(t)$ by setting proper $p(t)$ regardless of the level of $\hat{k}(t)$, if there is no supply side constraint. Based on this fact, we derive the following two propositions from the lemma 1.

Proposition 2: Operational principle of controlling water price

If there are no supply side constraints on setting the water price such as water production capacity limitation, the government can achieve any desirable growth rate of the private capital stock through controlling the water price based on the following operational principle.

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[13] We exclude $\xi = 0$, which requires either k or p is zero, from the domain.

$$\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} \begin{matrix} > \\ = \\ < \end{matrix} 0, \quad \text{if and only if} \quad p(t) \begin{matrix} < \\ = \\ > \end{matrix} \xi^{-\frac{1-\beta_Q}{\beta_Q}} \left\{\hat{k}(t)\right\}^{-\frac{\beta_L}{\beta_Q}}.$$

Proof: Since we can freely change $\xi(t)$ by setting proper $p(t)$ regardless of the level of $\hat{k}(t)$, the lemma 1 guarantees that we can control the growth rate of $\hat{k}(t)$ from the negative infinity to the positive infinity by choosing $\xi(t)$ through setting $p(t)$, regardless of the level of $\hat{k}(t)$.

Q.E.D.

Proposition 3: Stability of the steady-state

If the water price is set at constant, $\hat{k}(t)$ and $\hat{c}(t)$ converge towards the steady-state.

Proof: Assume that the constant water price is such that $\xi(t) < \bar{\xi}$. Then the lemma 1 tells that $\hat{k}(t)$ grows at positive rate and consequently $\xi(t)$ increases towards $\bar{\xi}$. And vice versa. Thus the steady-state is globally stable within the domain of $\phi^k(\xi)$. Since $\hat{c}(t)$ is a function of $\hat{k}(t)$ and $p(t)$ only, at this steady-state $\hat{c}(t)$ becomes constant.

Q.E.D.

Finally we derive the following proposition from Eq. (9) and the lemma 1.

Proposition 4: Steady-state optimal consumption

If there are no supply side constraints on setting the water price, the government can induce any desirable level of the optimal households' consumption by setting appropriate constant water price.

Proof: Recall that the steady-state level of the optimal consumption is given by Eq. (9) as

$$\hat{c} = \frac{\hat{k}\left(b_{10}\bar{\xi}^{-1} - b_9\right)}{b_1 p^{1-\varphi}}\left(1 + \frac{\beta_L}{\beta_K - b_{11}\bar{\xi}}\right) = b_{15}\, p^{-\left(1-\varphi+\frac{\beta_Q}{\beta_L}\right)}, \text{ where}$$

$$b_{15} \equiv \varphi^{\varphi}(1-\varphi)^{1-\varphi}(\delta+\rho)^{\frac{\beta_Q-1}{\beta_L}}\beta_K^{\frac{\beta_K}{\beta_L}}\beta_Q^{\frac{\beta_Q}{\beta_L}}\{(\rho-\nu)\beta_K + (\delta+\rho)\beta_L\} > 0.$$

Since the exponent of p is not zero but strictly negative, any positive steady-state optimal level of $\hat{c}$ can be induced by setting a proper constant price of water.

Q.E.D.

These three propositions might be of policy makers' interest. For instance, if the economy is free from the water scarcity problems and the government has discretion in setting the water price, the propositions 3 and 4 tell that the government can induce any desirable household consumption level by setting the water price based on the equation in the latter proposition. Needless to say, these "desirable" results are largely due to the strong assumptions such as full employment or full market equilibrium which are rarely found in the real world, above all in the developing economies. Nevertheless, these propositions provide a useful benchmark based on which we can investigate the outcome of relaxing each assumption. Moreover, they facilitate analysis of supply side problems which are particularly important in the water-scarce developing economies.

4. Second-stage Optimisation

4.1 Water production

As in the case of private goods production, we drastically simplify the actual processes consisting of water production, e.g. harnessing raw water from the natural hydrological cycle, water purification, transmission, and so forth into the aggregate water production function $Q = F^W(G)$, where Q is an aggregate water production and G is an aggregate public capital stock.

Weitzman (1970) argues that social overhead capital including sanitation facilities, irrigation and drainage facilities, and water supply facilities are belonging to "the β sector" characterised by very high capital intensity. This might justify the above specification. Note that here we use an aggregate production function because water constraint is manifested not in per capita term but in absolute term.

Another basic feature of the β sector discussed by Weitzman (1970) is substantial economies of scale due to indivisibility and cost lumpiness in this sector. This is perfectly relevant to the case of water production, which is often associated with large-scale facilities such as dams, treatment plants and pipelines (Young and Haveman 1985). Hence the shape of the water production function might not be smooth but kinked at several points as illustrated in Figure 1.

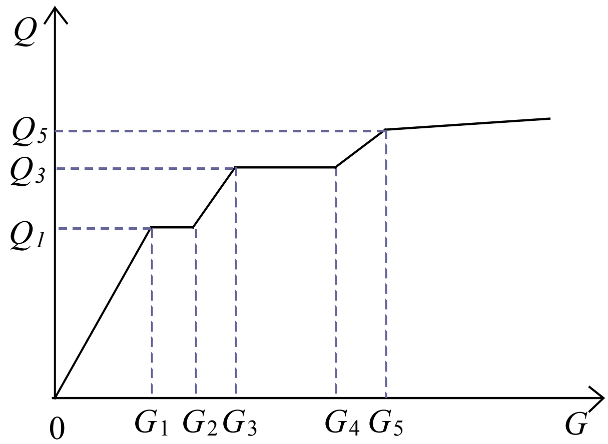

Figure 1 Conceptual illustration of water production function

Here some clarification may be necessary. In our model setting it is necessary to interpret this function as the relationship between public capital stock and capacity to produce clean water *on condition that water sustainability is not endangered*. Thus the necessary public capital covers not only narrowly defined water supply costs but also wastewater management costs indispensable to maintain water sustainability. On this ground our model sets aside the water quality issue.[14]

When the supplied amount of water is smaller than Q_1, harnessing water from hydrological cycle is technically easy and water related capital might be divisible. To supply more than Q_1 we might need to install some large-scale facility, e.g. a large dam, of which construction requires certain amount of capital corresponding to G_2 - G_1. Only after the installation of this facility it is possible to increase water supply capacity up to Q_3 by further capital accumulation (section between G_2 and G_3). In

[14] Though analytical tractability and logical coherence is maintained by this assumption, an empirical applicability of the model is significantly reduced. An explicit treatment of water quality is a formidable but urgent issue to be tackled in the future.

other words, the marginal product of water with respect to G is zero between G_1 and G_2 (i.e. dF^W/dG =0). After repeating this process several times we will reach certain upper limit of water quantity (Q_5) harnessed from natural hydrological cycle without violating sustainability of ecosystem. Though it is possible to increase water supply capacity above this quantity by introducing water recycling, desalinisation of sea water, water import, and so on, it is highly likely that the marginal water product of these technologies be very low, as is shown in Figure 1. Note that we assume the water production function is twice continuously differentiable at any point for the analytical purpose.

The water balance constraint can be expressed as

$$N_0 e^{vt}(q_H + q_M) \leq F^W(G), \tag{11}$$

where N_0 is the initial population.

The left hand side is an aggregate water demand, while the right hand side is the water supply capacity.

4.2 Government

It is assumed that the budget neutral government collects volumetric water charges from both households and firms and it spends all the collected charges for public capital investment (I^G).

$$p(q_H + q_M)N_0 e^{vt} = I^G. \tag{12}$$

The assumption that the government undertakes water service is not only justifiable considering the natural monopolistic feature of water provision but also realistic in most developing countries. In the context of closed economy assumption the assumption of budget neutral government is sensible. From this assumption the equation of motion of the public capital becomes

$$\dot{G} = I^G - \delta G = p(q_H + q_M)N_0 e^{vt} - \delta G. \tag{13}$$

The aim of the government is maximising the “intra-generational” social welfare of the current generation by choosing water price with observing the sustainability condition. Since we assume identical households, the government problem can be expressed as

$$\underset{p}{Max}\int_0^\infty e^{-(\rho-\nu)t}\frac{\hat{c}\left(\hat{k},p\right)^{(1-\sigma)}}{1-\sigma}dt \text{ , subject to} \tag{14a}$$

$$\dot{G}=p\left\{\hat{q}_H\left(\hat{k},p\right)+\hat{q}_M\left(\hat{k},p\right)\right\}N_0e^{\nu t}-\delta G\text{ , } G(0)=G_0, \tag{14b}$$

$$\dot{\hat{k}}=\hat{k}\phi^k\left(\hat{k},p\right),\ \hat{k}(0)=k_0,\text{ and} \tag{14c}$$

$$F^W(G)-\left\{\hat{q}_H\left(\hat{k},p\right)+\hat{q}_M\left(\hat{k},p\right)\right\}N_0e^{\nu t}\geq 0\text{ .} \tag{14d}$$

The corresponding Lagrangian consisting of the current value Hamiltonian and the water balance constraint is

$$L^G=\frac{\hat{c}\left(\hat{k},p\right)^{(1-\sigma)}}{1-\sigma}+\mu^G\left[p\left\{\hat{q}_H\left(\hat{k},p\right)+\hat{q}_M\left(\hat{k},p\right)\right\}N_0e^{\nu t}-\delta G\right]+\mu^k\hat{k}\phi^k\left(\hat{k},p\right)$$
$$+\Theta\left[F^W(G)-\left\{\hat{q}_H\left(\hat{k},p\right)+\hat{q}_M\left(\hat{k},p\right)\right\}N_0e^{\nu t}\right],$$

where μ^G, μ^k and Θ are the Lagrange multipliers associated with G, $\hat{k}$ and the water constraint, respectively.

Assuming an interior solution, the necessary and sufficient conditions are as follows.[15]

$$\frac{\partial L^G}{\partial p}=0\ \Rightarrow\ \hat{c}^{-\sigma}\frac{\partial\hat{c}}{\partial p}+\mu^G(\hat{q}_H+\hat{q}_M)+\left(\mu^Gp-\Theta\right)\left(\frac{\partial\hat{q}_H}{\partial p}+\frac{\partial\hat{q}_M}{\partial p}\right)N_0e^{\nu t}$$
$$+\mu^k\hat{k}\frac{\partial\phi^k}{\partial p}=0 \tag{15a}$$

$$\dot{\mu}^G-(\rho-\nu)\mu^G=-\frac{\partial L^G}{\partial G}\ \Rightarrow\ \frac{\dot{\mu}^G}{\mu^G}=\rho+\delta-\nu-\frac{\partial F^W}{\partial G}\frac{\Theta}{\mu^G} \tag{15b}$$

$$\dot{\mu}^k-(\rho-\nu)\mu^k=-\frac{\partial L^G}{\partial\hat{k}}\ \Rightarrow\ \dot{\mu}^k=\left(\rho-\nu-\phi^k-\hat{k}\frac{\partial\phi^k}{\partial\hat{k}}\right)-\hat{c}^{-\sigma}\frac{\partial\hat{c}}{\partial\hat{k}}$$
$$-\left(\mu^Gp-\Theta\right)\left(\frac{\partial\hat{q}_H}{\partial\hat{k}}+\frac{\partial\hat{q}_M}{\partial\hat{k}}\right)N_0e^{\nu t} \tag{15c}$$

[15] This problem satisfies the Mangasarian Sufficiency Theorem (1966). See footnote 8.

The Kuhn-Tucker condition for the water constraint is

$$\Theta \geq 0,\; F^W - (\hat{q}_H + \hat{q}_M)N_0 e^{\nu t} \geq 0,\; \Theta\left\{F^W - (\hat{q}_H + \hat{q}_M)N_0 e^{\nu t}\right\} = 0. \tag{15d}$$

Note that Θ is zero only if the water constraint is satisfied with strict inequality. The transversality conditions are

$$\lim_{t\to\infty}\left[e^{-(\rho-\nu)t}\mu^G(t)\cdot G(t)\right] = 0,\text{ and} \tag{15e}$$

$$\lim_{t\to\infty}\left[e^{-(\rho-\nu)t}\mu^k(t)\cdot \hat{k}(t)\right] = 0. \tag{15f}$$

4.3 The optimal trajectories

It is known that each Lagrange multiplier represents the shadow price of its corresponding constraint. Since G and $\hat{k}$ are the same good, the ratio of their shadow prices in aggregate term is unity along the optimal trajectories. Otherwise it is possible to achieve higher social welfare by allocating more capital good to invest in either capital with higher shadow price. Hence the optimality requires $\mu^G = \frac{\mu^k}{N_0 e^{\nu t}} \equiv \mu$.

The necessary conditions for the optimal trajectories are depending upon whether water supply capacity exceeds water demand (i.e. $\Theta = 0$) or not. These two cases are separately analysed.

(1) Case 1: $\Theta = 0$ (Water supply capacity exceeds the demand)

In this case the optimality condition (15b) determines the optimal value of μ as $\mu(t) = \mu(0)e^{(\delta-\nu+\rho)t}$. By putting this and $\Theta = 0$ into another optimality condition (15a) we have

$$\hat{c}^{-\sigma}\frac{\partial \hat{c}}{\partial p} = -\mu(0)N_0 e^{(\delta+\rho)t}\left\{(\hat{q}_H + \hat{q}_M) + p\left(\frac{\partial \hat{q}_H}{\partial p} + \frac{\partial \hat{q}_M}{\partial p}\right) + \hat{K}\frac{\partial \phi^k}{\partial p}\right\}, \tag{16}$$

where $\hat{K} \equiv \hat{k}N_0 e^{\nu t}$: economy wide aggregate private capital stock.

By taking time derivative of the both hand sides of Eq. (16) with logarithmic transformation, we obtain

$$\frac{d}{dt}\ln\left(\hat{c}^{-\sigma}\hat{c}_p\right)$$

$$=\frac{d}{dt}\ln\left[-\mu(0)N_0e^{(\delta+\rho)t}\left\{(\hat{q}_H+\hat{q}_M)+p\left(\frac{\partial\hat{q}_H}{\partial p}+\frac{\partial\hat{q}_M}{\partial p}\right)+\hat{K}\frac{\partial\phi^k}{\partial p}\right\}\right]. \tag{17}$$

The left hand side of (17) becomes

$$\frac{d}{dt}\ln\left(\hat{c}^{-\sigma}\hat{c}_p\right)=\left(\varepsilon_{pp}-\sigma\varepsilon_p\right)\frac{\dot{p}}{p}+\left(\varepsilon_{pk}-\sigma\varepsilon_k\right)\frac{\dot{\hat{k}}}{\hat{k}}, \text{ where} \tag{18}$$

$\varepsilon_p\equiv\frac{\partial\hat{c}}{\partial p}\frac{p}{\hat{c}}$, $\varepsilon_{pp}\equiv\frac{\partial\hat{c}_p}{\partial p}\frac{p}{\hat{c}_p}$, $\varepsilon_k\equiv\frac{\partial\hat{c}}{\partial\hat{k}}\frac{\hat{k}}{\hat{c}}$, and $\varepsilon_{pk}\equiv\frac{\partial\hat{c}_p}{\partial\hat{k}}\frac{\hat{k}}{\hat{c}_p}$: the elasticities of consumption and marginal consumption with respect to price or private capital stock.

The right hand side of (17) becomes

$$\frac{d}{dt}\ln\left[-\mu(0)N_0e^{(\delta+\rho)t}\left\{(\hat{q}_H+\hat{q}_M)+p\left(\frac{\partial\hat{q}_H}{\partial p}+\frac{\partial\hat{q}_M}{\partial p}\right)+\hat{K}\frac{\partial\phi^k}{\partial p}\right\}\right]$$

$$=(\delta+\rho)+\frac{d}{dt}\ln\left\{(\varepsilon_p+1-\varphi)\hat{q}_H-b_6\hat{q}_M+N_0e^{\nu t}\hat{k}\frac{\partial\phi^k}{\partial p}\right\}$$

$$=\frac{(\delta+\rho)\{(\varepsilon_p+1-\varphi)\hat{q}_H-b_6\hat{q}_M\}+(\delta+\nu+\rho)N_0e^{\nu t}\hat{k}\phi_p^k}{(\varepsilon_p+1-\varphi)\hat{q}_H-b_6\hat{q}_M+N_0e^{\nu t}\hat{k}\phi_p^k}$$

$$+\frac{\{(1-\varphi)(2\varepsilon_p-\varphi)+\varepsilon_p\varepsilon_{pp}\}\hat{q}_H+b_6\hat{q}_M\big/(1-\beta_Q)+N_0e^{\nu t}\hat{k}\phi_p^k\varepsilon_p^{\phi_p^k}}{(\varepsilon_p+1-\varphi)\hat{q}_H-b_6\hat{q}_M+N_0e^{\nu t}\hat{k}\phi_p^k}\left(\frac{\dot{p}}{p}\right)$$

$$+\frac{\varepsilon_k(\varepsilon_{kp}+1-\varphi)\hat{q}_H-b_6b_7\hat{q}_M+N_0e^{\nu t}\hat{k}\phi_p^k\left(1+\varepsilon_k^{\phi_p^k}\right)}{(\varepsilon_p+1-\varphi)\hat{q}_H-b_6\hat{q}_M+N_0e^{\nu t}\hat{k}\phi_p^k}\left(\frac{\dot{\hat{k}}}{\hat{k}}\right), \tag{19}$$

where $\varepsilon_{kp}\equiv\frac{\partial\hat{c}_k}{\partial p}\frac{p}{\hat{c}_k}$, $\phi_p^k\equiv\frac{\partial\phi^k}{\partial p}$, $\varepsilon_p^{\phi_p^k}\equiv\frac{\partial\phi_p^k}{\partial p}\frac{p}{\phi_p^k}$, and $\varepsilon_k^{\phi_p^k}\equiv\frac{\partial\phi_p^k}{\partial\hat{k}}\frac{\hat{k}}{\phi_p^k}$.

From (18) and (19) we obtain the differential equation of the water price.

$$\frac{\dot{p}}{p} = \phi_1^p\left(\hat{k}, p\right) \equiv \frac{B_1\left(\hat{k}, p\right) + B_2\left(\hat{k}, p\right)\phi^k\left(\hat{k}, p\right)}{B_3\left(\hat{k}, p\right)}, \text{ where} \tag{20}$$

$$B_1\left(\hat{k}, p\right) \equiv (\delta + \rho)\left\{\left(\varepsilon_p + 1 - \varphi\right)\hat{q}_H - b_6\hat{q}_M + \frac{\delta + \nu + \rho}{\delta + \rho} N_0 e^{\nu t}\hat{k}\phi_p^k\right\},$$

$$B_2\left(\hat{k}, p\right) \equiv \left\{(1 - \varphi)\left(\varepsilon_k - \varepsilon_{pk}\right) - \sigma\varepsilon_k\left(\varepsilon_p + 1 - \varphi\right)\right\}\hat{q}_H$$

$$+ b_6\left(\varepsilon_{pk} - \sigma\varepsilon_k - b_7\right)\hat{q}_M + N_0 e^{\nu t}\hat{k}\phi_p^k\left(1 + \varepsilon_k^{\phi_p^k}\right), \text{ and}$$

$$B_3\left(\hat{k}, p\right) \equiv \left\{(1 - \varphi)\left(\varepsilon_{pp} - 2\varepsilon_p + \varphi\right) - \sigma\varepsilon_p\left(\varepsilon_p + 1 - \varphi\right)\right\}\hat{q}_H$$

$$- b_6\left(\varepsilon_{pp} - \sigma\varepsilon_p + \frac{1}{1 - \beta_Q}\right)\hat{q}_M - N_0 e^{\nu t}\hat{k}\phi_p^k\varepsilon_p^{\phi_p^k}.$$

The optimal trajectories are determined by the following system.

$$\frac{\dot{p}}{p} = \phi_1^p\left(\hat{k}, p\right), \ p(0) \text{ is free, and} \tag{21a}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = \phi^k\left(\hat{k}, p\right), \ \hat{k}(0) = k_0 \text{ given.} \tag{21b}$$

This system does not contain the differential equation of *G* and the stock level of *G* is unilaterally determined by (14b).

The transversality condition implies the following proposition.

Proposition 5: Transient nature of the optimal path with $\Theta = 0$

The trajectories determined by the system (21a) and (21b) are optimal only in the short-run due to the fact that they cannot satisfy the transversality condition (15d).

Proof: By taking integral of the both hand sides of the equation of motion of public capital, we obtain

$$G(t) = G_0 e^{-\delta t} + N_0 e^{-\delta t}\int_0^t p(s)\{\hat{q}_H(s) + \hat{q}_M(s)\}e^{(\delta+\nu)s}ds$$

Putting this and $\mu(t) = \mu(0)e^{(\delta-\nu+\rho)t}$ into the transversality condition, we have

$$\lim_{t\to\infty}\left[\mu(0)e^{\delta t}\cdot\left\{G_0e^{-\delta t}+N_0e^{-\delta t}\int_0^t p(\hat{q}_H+\hat{q}_M)e^{(\delta+\nu)s}ds\right\}\right]$$
$$=\mu(0)G_0+\mu(0)N_0\lim_{t\to\infty}\left[\int_0^t p(\hat{q}_H+\hat{q}_M)e^{(\delta+\nu)s}ds\right]=0.$$

As the integrand is always non-negative, this condition cannot be satisfied unless $\mu(0)$ is zero which requires either zero consumption or zero marginal consumption with respect to price from (15a). The former cannot be the socially optimal while the latter case seems a trivial exceptional case where water pricing does not affect water demand. On this rationale the case $\mu(0) = 0$ is precluded from the analysis. Hence the trajectories determined by the system (21a) and (21b) cannot satisfy the transversality condition.

Q.E.D.

(2) Case 2: $\Theta > 0$ (Water supply capacity equals to the demand)

When $\Theta > 0$ we have the following equality.

$$F^W=(\hat{q}_H+\hat{q}_M)N_0e^{\nu t} \tag{22}$$

In this case $\Theta = p\mu$ holds, since the shadow price of relaxing water balance consumption means the social benefit of providing additional water of which relative price to the capital good is p.

With $\Theta = p\mu$ and Eq. (11), the necessary and sufficient conditions (15a) and (15c) are transformed into

$$\hat{c}^{-\sigma}\frac{\partial\hat{c}}{\partial p}+\mu\left(F^W+N_0e^{\nu t}\hat{k}\phi_p^k\right)=0\text{, and} \tag{23a}$$

$$\frac{\dot{\mu}}{\mu}=\delta-\nu+\rho-p\frac{dF^W}{dG}. \tag{23b}$$

With the familiar technique the following equation are derived from (23a).

$$\frac{\dot{\mu}}{\mu} = \left\{ \left(\varepsilon_{pp} - \sigma\varepsilon_p \right) - \frac{\varepsilon_p^{\phi_p^k} \hat{K}\phi_p^k}{F^W + \hat{K}\phi_p^k} \right\} \frac{\dot{p}}{p} + \left\{ \left(\varepsilon_{pk} - \sigma\varepsilon_k \right) - \frac{\left(1 + \varepsilon_k^{\phi_p^k} \right) \hat{K}\phi_p^k}{F^W + \hat{K}\phi_p^k} \right\} \frac{\dot{\hat{k}}}{\hat{k}}$$

$$- \frac{F^W \varepsilon_G}{F^W + \hat{K}\phi_p^k} \frac{\dot{G}}{G} - \frac{\nu \hat{K}\phi_p^k}{F^W + \hat{K}\phi_p^k}, \tag{24}$$

where $\varepsilon_G \equiv \frac{dF^W}{dG} \frac{G}{F^W}$: the elasticity of the water production.

By substituting Eq. (22) into Eq. (14b) the following differential equation of G is obtained.

$$\frac{\dot{G}}{G} = \frac{pF^W}{G} - \delta \equiv \phi^G(G, p), \; G(0) = G_0 \text{ (given)}. \tag{25}$$

From equations (23b), (24) and (25) the differential equation of p is derived as

$$\frac{\dot{p}}{p} = \frac{F^w \{(1 - \varepsilon_G)\delta - \nu + \rho\} + N_0 e^{\nu t} \hat{k}\phi_p^k \{(1 - \varepsilon_G)\delta + \rho - \varepsilon_G \phi^G\}}{\left(F^w + N_0 e^{\nu t} \hat{k}\phi_p^k \right)\left(\varepsilon_{pp} - \sigma\varepsilon_p \right) - \varepsilon_p^{\phi_p^k} N_0 e^{\nu t} \hat{k}\phi_p^k}$$

$$- \frac{\left(F^w + N_0 e^{\nu t} \hat{k}\phi_p^k \right)\left(\varepsilon_{pk} - \sigma\varepsilon_k \right) - \left(1 + \varepsilon_k^{\phi_p^k} \right) N_0 e^{\nu t} \hat{k}\phi_p^k}{\left(F^w + N_0 e^{\nu t} \hat{k}\phi_p^k \right)\left(\varepsilon_{pp} - \sigma\varepsilon_p \right) - \varepsilon_p^{\phi_p^k} N_0 e^{\nu t} \hat{k}\phi_p^k} \phi^k \equiv \phi_2^p\left(\hat{k}, G, p \right). \tag{26}$$

The optimal trajectories are determined by the following system of the differential equations.

$$\frac{\dot{G}}{G} = \phi^G(G(t), p(t)), \; G(0) = G_0 \text{ (given)}, \tag{27a}$$

$$\frac{\dot{p}}{p} = \phi_2^p\left(\hat{k}(t), G(t), p(t) \right), \; p(0) \text{ is free, and} \tag{27b}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = \phi^k\left(\hat{k}(t), p(t) \right), \; \hat{k}(0) = k_0 \text{ given}. \tag{27c}$$

The transversality condition imposes the following condition to have the long-run optimal paths.

Proposition 6: Condition to have the long-run optimal paths

There exist the long-run optimal time paths if and only if the stock of public capital $G(t)$ satisfies $\varepsilon_G(G(t))>1$ as t goes to infinity, along with the trajectories determined by the system (27a), (27b) and (27c).

Proof : The proposition 5 proves that the long-run optimal time paths must be determined by the system (27a), (27b) and (27c). The condition for such time paths to be the long-run optimal is to satisfy the transversality condition (15d) which requires that the rate of change of $e^{-(\rho-\nu)t}\mu(t)\cdot G(t)$ is kept negative as t goes to infinity. It means that along with the trajectories determined by the system (27a), (27b) and (27c) the following inequality has to hold as t goes to infinity.

$$\frac{d}{dt}\ln\left\{e^{-(\rho-\nu)t}\mu(t)\cdot G(t)\right\}=-(\rho-\nu)+\frac{\dot{\mu}}{\mu}+\frac{\dot{G}}{G}=\frac{pF^W}{G}(1-\varepsilon_G)<0$$

The above condition requires $\varepsilon_G(G(t))>1$ as t goes to infinity.

Q.E.D.

Recall the general shape of the water production function. We may need to keep the stock of public capital less than a certain level such that the necessary condition in the proposition 6 is satisfied. This requires non-positive growth rate of G in the long run. The proposition 6 rules out an infinite growth of water production capacity, which consequently precludes the possibility to maintain non-declining per capita consumption with positive population growth for the infinite time.

5. Qualitative Analysis of the Optimal Time Paths

The raison d'etre of the analytic model lies in its ability to investigate qualitative properties of the optimal trajectories such as stability of the singular point (steady-state) and structural stability of the system. For such an investigation it is necessary to find the optimal steady state(s) of the system.

The proposition 5 dictates that the optimal steady-state(s) must be associated with Θ> 0. Hence the optimal steady-state(s) of the system must satisfy the following system of equations.

$$\phi^G = \frac{pF^W}{G} - \delta = 0 \tag{28a}$$

$$\phi_2^p = \frac{F^w\{(1-\varepsilon_G)\delta - \nu + \rho\} + N_0 e^{\nu t}\hat{k}\phi_p^k\{(1-\varepsilon_G)\delta + \rho\}}{\left(F^w + N_0 e^{\nu t}\hat{k}\phi_p^k\right)\left(\varepsilon_{pp} - \sigma\varepsilon_p\right) - \varepsilon_p^{\phi_p^k} N_0 e^{\nu t}\hat{k}\phi_p^k} = 0 \tag{28b}$$

$$\phi^k = \frac{b_{12}\left(p^{b_6}\hat{k}^{b_8}\right)^2 - b_{13}p^{b_6}\hat{k}^{b_8} + b_{14}}{\sigma\left(\beta_K - b_{11}p^{b_6}\hat{k}^{b_8}\right)p^{b_6}\hat{k}^{b_8}} = 0 \tag{28c}$$

Let G^*, p^* and $\hat{k}^*$ denote the optimal values of referent variables and $\overline{G}$, $\overline{p}$ and $\overline{k}$ denote their optimal steady-state values.

Proposition 7: Optimal steady-state

Along the optimal trajectories determined by the system (27a), (27b) and (27c), there exists at least one optimal steady-state if and only if

(a) The population growth rate ν is 0, and

(b) G^* is such that $\varepsilon_G(G^*) = 1 + \dfrac{\rho}{\delta}$.

Proof : It is clear from the water sustainability condition (22) that the optimal steady-state could exist only if ν = 0. When this condition is satisfied, Eq. (28b) becomes $\dfrac{(1-\varepsilon_G)\delta + \rho}{\left(\varepsilon_{pp} - \sigma\varepsilon_p\right) - \varepsilon_p^{\phi_p^k}\left\{1 + F^W / \left(\hat{K}\phi_p^k\right)\right\}^{-1}} = 0$. Assuming that the denominator is non-zero, this condition is satisfied if and only if $\varepsilon_G = 1 + \dfrac{\rho}{\delta}$. This equation, which also satisfies the transversality condition from the proposition 5, determines the optimal steady-state level of public capital stock $\overline{G}$. Subsequently Eq. (28a) determines the optimal steady-state price $\overline{p}$ as $\overline{p} = \dfrac{\delta \overline{G}}{F^W(\overline{G})}$ and finally the optimal steady-state stock level of private capital $\overline{k}$ is given as $\overline{k} = \xi^{-\frac{1-\beta_Q}{\beta_L}} (\overline{p})^{-\frac{\beta_Q}{\beta_L}}$.

Q.E.D.

Recall the shape of water production function in Figure 1. There could be more than one values of *G* satisfying the condition (b) in the proposition 7. It means that there could exist more than one optimal steady-states depending on the shape of the water production function.

Due to nonlinearity of the original system defined by Equations (27a), (27b) and (27c) it is difficult to analyse global stability or instability of the optimal steady-state. If the optimal steady-state is partially stable, which is frequently observed in the optimal growth models, the Liapunov's second method does not work (Gandolfo 1997). In fact an application of the second method with a Euclidean distance function as a candidate of Liapunov function results that the time derivative of this candidate is neither positive nor negative definitive, which is consistent with partial stability. Hence we focus on analysing the local stability of the optimal steady state by linearisation method.

As this system is autonomous, the following linearised system near steady state is certainly a uniformly good approximation to the original nonlinear system (27a) - (27c) around the optimal steady state (Gandolfo 1997).

$\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}^e)$, where $\boldsymbol{x} \equiv \begin{bmatrix} G \\ p \\ \hat{k} \end{bmatrix}$, $\boldsymbol{x}^e \equiv \begin{bmatrix} \overline{G} \\ \overline{p} \\ \overline{k} \end{bmatrix}$, and $\boldsymbol{A}$ is the Jacobian matrix of the original system evaluated at the optimal steady state, i.e.

$$\boldsymbol{A} \equiv \begin{bmatrix} \dfrac{\partial G\phi^G}{\partial G} & \dfrac{\partial G\phi^G}{\partial p} & \dfrac{\partial G\phi^G}{\partial \hat{k}} \\ \dfrac{\partial p\phi_2^P}{\partial G} & \dfrac{\partial p\phi_2^P}{\partial p} & \dfrac{\partial p\phi_2^P}{\partial \hat{k}} \\ \dfrac{\partial \hat{k}\phi^k}{\partial G} & \dfrac{\partial \hat{k}\phi^k}{\partial p} & \dfrac{\partial \hat{k}\phi^k}{\partial \hat{k}} \end{bmatrix}_{(\boldsymbol{x} = \boldsymbol{x}^e)} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \text{ where}$$

$A_{11} \equiv \rho > 0$, $A_{12} \equiv \dfrac{\delta\overline{G}}{\overline{p}} > 0$,

$$A_{21} \equiv \frac{\overline{p}}{D_1}\left\{ \frac{\rho(\delta+\rho)}{\delta\overline{G} - b_6 b_{16}\overline{K}} - \overline{p}\frac{d^2 F^W}{dG^2}\bigg|_{\overline{G}} \right\},$$

$$A_{22} \equiv \frac{b_6 b_{16}}{D_1}\left[\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k - \left\{ b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \frac{\delta+\rho}{b_6 b_{16}} \right\}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}} \right)^{-1} \right],$$

$$A_{23} \equiv \frac{b_8 b_{16}\overline{p}}{D_1\overline{k}}\left[\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k - \left\{ b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} \right\}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}} \right)^{-1} \right],$$

$A_{32} \equiv -b_6 b_{16}\dfrac{\overline{k}}{\overline{p}} < 0$, and $A_{33} \equiv -b_8 b_{16} < 0$,

in which $b_{16} \equiv \dfrac{(\delta+\rho)\{\rho\beta_K + (\delta+\rho)\beta_L\}}{\rho\sigma\beta_K} > 0$,

$b_{17} \equiv 2 - \dfrac{\delta}{\rho} + \dfrac{2\delta\beta_K\beta_Q}{(1-\beta_Q)\{\rho\beta_K + (\delta+\rho)\beta_L\}}$, and

$$D_1 \equiv \overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p + b_{17}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}} \right)^{-1}.$$ [16]

The following two propositions summarise the local stability of the optimal trajectories around the optimal steady-state(s).

Proposition 8: Structural instability of the optimal steady-state

Along the optimal trajectories determined by the system (27a), (27b) and (27c), their exist bifurcations (threshold parameter values) at which qualitative changes in the local stability properties of the optimal steady-state(s) occur.[17]

Proof : The local stability properties of the linearised system around the optimal steady-state are partially determined by the signs of the determinant and the trace of the matrix $\boldsymbol{A}$ (Gandolfo 1997). To prove the existence of bifurcation it is enough to show that the sign of determinant of the matrix $\boldsymbol{A}$, denoted as det $\boldsymbol{A}$, depends on

[16] For the derivation of the matrix $\boldsymbol{A}$, see Appendix A3.

parameter values. By its definition and due to the fact that $A_{13}=A_{31} = 0$, $\det \boldsymbol{A} = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}A_{21}A_{33}$, which is evaluated, after some algebraic manipulation, as $\frac{b_8 b_{16}}{D_1}\left\{\rho(\delta+\rho) - \delta\overline{p}\overline{G}\left.\frac{d^2F^W}{dG^2}\right|_{\overline{G}}\right\}$. Due to the assumption of diminishing marginal water product, the sign of det $\boldsymbol{A}$ depends on the sign of D_1. As shown in Appendix 3, the sign of D_1 is indeterminate unless parameter values are given. As a result, the sign of det $\boldsymbol{A}$ is affected by parameter values as well as the optimal steady-state values.

Q.E.D.

Due to this structural instability it is necessary to set parameter values for studying the local stability properties. The following parameter values are chosen as the base case.

- σ (elasticity of marginal felicity): 1.7
- φ (weight of market good consumption in satisfaction production): 0.6
- δ (depreciation rate): 0.05
- ρ (rate of pure time preference): 0.05
- β_K (factor share of private capital in market commodity production): 0.4
- β_Q (factor share of water in market commodity production): 0.3
- β_L (factor share of labour in market commodity production): 0.3

Proposition 9: Local stability of optimal steady-state in base case

Assuming the base case parameter values, the matrix $\boldsymbol{A}$ has always one unstable and two stable eigenvalues regardless of the number of the optimal steady- states. Hence it is possible to put the economy on a stable manifold towards one of the optimal steady-states by choosing proper initial value of the rate of water charge $p(0)$.

Proof: See Appendix A4.

It is difficult to assert the robustness of the proposition 9 because it depends on the combination of parameter values. Moreover, except for few cases including the base

[17] The definition of bifurcation is from Gandolfo (1997).

case, the local stability conditions involve not only parameter values but also the optimal steady-state values. Nevertheless, an unsubtle sensitivity analysis, in which each of parameter values alone is changed from the base case values, was conducted to illustrate its robustness. The results of sensitivity analysis are shown in Table 1.

Table 1 Results of Sensitivity Analysis

Parameter	Valid range	Remark
σ	≥ 1.62	If $\sigma \leq 1.61$ the sign of D_1 is indeterminate unless the optimal values are obtained.
φ	≤ 0.65	If $\varphi \geq 0.66$ the sign of D_1 is indeterminate unless the optimal values are obtained.
δ	≤ 0.05	If $\delta \geq 0.06$ the sign of D_1 is indeterminate unless the optimal values are obtained.
ρ	≥ 0.05	If $\rho \leq 0.04$ the sign of D_1 is indeterminate unless the optimal values are obtained.
$\beta_L \mid \beta_K = 0.4$	Entire range	–
$\beta_L \mid \beta_Q = 0.3$	≤ 0.33	If $\beta_L \geq 0.34$ the sign of D_1 is indeterminate unless the optimal values are obtained.

It can be observed that the proposition 9 is sensitive to the parameter values, in particular to the depreciation rate δ considering the fact that some authors employ the value higher than 0.05 for δ. For σ and ρ the valid range may cover the values reported in the existing literature.[18] Though the base case parameter values are quite plausible, the proposition 9 seems quite sensitive to the parameter values. For the parameter values outside the valid range shown in Table 1 the local stability conditions depends not only on the parameter values but also on the optimal steady-state values, and the local stability conditions may vary among optimal steady-states if there are more than one optimal steady-state.

6. Conclusion

In this chapter the analytic model is developed based on RCK growth model with introducing continuous monitoring-feedback in the households' price expectation

[18] See Footnote 6 of this chapter. Ostry and Reinhart (1992) report empirical estimates of σ and ρ for each of African, Asian and Latin American countries, in which the African average is 2.26 for σ and 0.064 for ρ based on the data of Morocco, Egypt, Ghana and Côte d'Ivoire between 1968 and mid-1980's.

formation as well as two-stage optimisation in which private optimisation and public optimisation processes are separated. These novel features improve the applicability of RCK growth model to quantitative policy analysis. This analytic model serves as a platform to develop the generalised "stylised" applied model which is a simple dynamic CGE model. The stylised applied model is explained in the next chapter.

The solution of the first-stage optimisation, i.e. the private optimisation, shows that if water is not scarce and the government can freely set the rate of water charge then the government can induce any desirable level of social welfare as the optimal steady-state by setting appropriate constant rate of water charge (Proposition 4).

The solution of the second-stage optimisation, i.e. the public optimisation, provides several implications.

Firstly, the long-run optimal trajectories exist only if the optimal level of public capital stock G^*satisfies $\varepsilon_G\left(G^*\right) \geq 1+\frac{\rho}{\delta}$, which rules out the possibility to keep eternally non-declining per capita consumption along the long-run optimal trajectories with population growth (Proposition 6). Note that the elasticity of water product with respect to public capital ε_G can be interpreted as a kind of water scarcity indicator in the sense that water abundant economy could easily have high ε_G. This result illuminates the importance of water scarcity issues in sustainable development.

Secondly, the optimal steady-state(s) exist if and only if population is constant and the optimal level of public capital stock G^*satisfies $\varepsilon_G\left(G^*\right) = 1+\frac{\rho}{\delta}$ (Proposition 7).

Thirdly, the local stability properties of the optimal steady state(s) depend on the parameter values (Proposition 8).

Lastly, with the base case parameter values the optimal steady state(s) exhibit saddle path stability associated with one unstable and two stable eigenvalues, under it is possible to realise the optimal steady-state(s) by choosing proper initial rate of water charge (Proposition 9).

These analytic results, mainly focusing on the neighbourhood of the optimal steady-state with constant population, are apparently of little relevance to underdeveloped economies as standalone but they provide useful benchmarks with which the results of more generalised model are evaluated.

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Mathematical Appendices

Appendix A1 Proof of Proposition 1

Recall the "clairvoyant" consumption function at period t.

$$c(t)=\eta(t)\left[m(t)+\int_t^{\infty} w(s)e^{-\int_t^s\{r(\tau)-\nu\}d\tau}ds\right], \tag{A1.1}$$

where $\eta(t)\equiv\left[b_1\{p(t)\}^{b_2}\int_t^{\infty}\{p(s)\}^{b_3}e^{\int_t^s\{b_4-b_5r(\tau)\}d\tau}ds\right]^{-1}$,

in which $b_1\equiv\varphi^{-\varphi}(1-\varphi)^{-(1-\varphi)}>0$, $b_2\equiv\dfrac{1-\varphi}{\sigma}>0$, $b_3\equiv\dfrac{(\sigma-1)(1-\varphi)}{\sigma}>0$,

$b_4\equiv\nu-\dfrac{\rho}{\sigma}$, and $b_5\equiv\dfrac{\sigma-1}{\sigma}>0$.

If we substitute true trajectories of exogenous variables w, r, and p with the households' expectation about them, which we assume constant at the current value at period t, we obtain

$$\int_t^{\infty} w(t)e^{-\int_t^s\{r(t)-\nu\}d\tau}ds=w(t)\int_t^{\infty}e^{-\{r(t)-\nu\}(s-t)}ds=w(t)\int_t^{\infty}\frac{d}{ds}\left[\frac{1}{-\{r(t)-\nu\}}e^{-\{r(t)-\nu\}(s-t)}\right]ds$$

$$=-\frac{w(t)}{\{r(t)-\nu\}}\left[e^{-\{r(t)-\nu\}(s-t)}\right]_t^{\infty}=\frac{w(t)}{\{r(t)-\nu\}}\left[1-\lim_{s\to\infty}e^{-\{r(t)-\nu\}s}\right], \tag{A1.2}$$

$$b_1\{p(t)\}^{b_2}\int_t^{\infty}\{p(t)\}^{b_3}e^{\int_t^s\{b_4-b_5r(t)\}d\tau}ds=b_1\{p(t)\}^{b_2+b_3}\int_t^{\infty}e^{\{b_4-b_5r(t)\}(s-t)}ds$$

$$=\frac{b_1\{p(t)\}^{1-\varphi}}{b_4-b_5r(t)}\left[e^{\{b_4-b_5r(t)\}(s-t)}\right]_t^{\infty}=\frac{b_1\{p(t)\}^{1-\varphi}}{b_5r(t)-b_4}\left[1-\lim_{s\to\infty}e^{\{b_4-b_5r(t)\}s}\right]. \tag{A1.3}$$

By substituting (A1.2) and (A1.3) into (A1.1), the consumption function becomes

$$c(t)=$$

$$\frac{b_5r(t)-b_4}{b_1\{p(t)\}^{1-\varphi}\left[1-\lim_{s\to\infty}e^{-\{b_5r(t)-b_4\}s}\right]}\left[m(t)+\left\{\frac{w(t)}{r(t)-\nu}\right\}\left\{1-\lim_{s\to\infty}e^{-\{r(t)-\nu\}s}\right\}\right] \tag{A1.4}$$

In order to converge two limits towards zero in (A1.4) we need the following two conditions.

$$r(t)-\nu > 0 \quad \Rightarrow \quad r(t) > \nu \text{, and} \qquad \text{(A1.5)}$$

$$b_5 r(t) - b_4 > 0 \;\Rightarrow\; r(t) > \frac{b_4}{b_5} = \frac{\nu\sigma - \rho}{\sigma - 1}. \qquad \text{(A1.6)}$$

First we check the relativity between two critical values as follows.

$$\nu - \frac{\nu\sigma-\rho}{\sigma-1} = \frac{\nu(\sigma-1)-(\nu\sigma-\rho)}{\sigma-1} = \frac{\rho-\nu}{\sigma-1} > 0. \qquad \text{(A1.7)}$$

Here recall that we previously assume that both the numerator and the denominator of the far right hand side are positive. Now we can examine the following 3 cases.

Case 1: $r > \nu$

Since (A1.5) and (A1.6) are satisfied, the above two limits converge to zero, and we obtain

$$c(t) = \frac{\{b_5 r(t) - b_4\}}{b_1 \{p(t)\}^{1-\varphi}} \left\{ m(t) + \frac{w(t)}{r(t)-\nu} \right\}. \qquad \text{(A1.8)}$$

Case 2: $b_4/b_5 < r \leq \nu$

If $r(t) = \nu$, $c(t)$ becomes positive infinity from (A1.1). Otherwise the limit corresponding to (A1.5) diverges to positive infinity and that of (A1.6) converges to zero. As a result, $c(t)$ becomes negative infinity.

Case 3: $r \leq b_4/b_5$

If $r(t) = b_4/b_5$, $c(t)$ becomes positive infinity from (A1.1). Otherwise, we need to modify (A1.4) into the following form to examine this case.

$$c = \frac{(b_5 r - b_4)m}{b_1 p^{1-\varphi}\left\{1 - \lim_{s\to\infty} e^{-(b_5 r - b_4)s}\right\}} + \frac{w}{b_1 p^{1-\varphi}(r-\nu)} \lim_{s\to\infty}\left\{\frac{1-e^{-(r-\nu)s}}{1-e^{-(b_5 r - b_4)s}}\right\} \qquad \text{(A1.9)}$$

Further, the far right limiting term is modified as

$$\lim_{s\to\infty}\left\{\frac{1-e^{-(r-v)s}}{1-e^{-(b_5 r-b_4)s}}\right\}=\lim_{s\to\infty}\left\{\frac{e^{(b_5 r-b_4)s}-e^{(b_5 r-b_4)s-(r-v)s}}{e^{(b_5 r-b_4)s}-1}\right\}=\frac{0-\lim_{s\to\infty}\exp\left\{\frac{1}{\sigma}(\rho-r)s\right\}}{0-1}$$

$$=\lim_{s\to\infty}\exp\left\{\frac{1}{\sigma}(\rho-r)s\right\}. \qquad \text{(A1.10)}$$

Since $r < v$ we have the inequality $\rho - r > \rho - v > 0$.

Hence the far right hand side of (A1.10) diverges to positive infinity. Since the first term of (A1.9) converges to zero and the second term diverges to positive infinity, $c(t)$ diverges to positive infinity.

In sum, we need the condition $r > v$ to have sensible consumption decision. When this condition is satisfied, the optimal consumption level is determined by (A1.8).

Q.E.D.

Appendix A2 Proof of Lemma 1

Recall

$$\phi^k(\xi)\equiv\frac{b_{12}\xi^2-b_{13}\xi+b_{14}}{\sigma(\beta_K-b_{11}\xi)\xi}\equiv\frac{\Omega(\xi)}{\sigma(\beta_K-b_{11}\xi)\xi} \qquad \text{(A2.1)}$$

for $0<\xi<\frac{\beta_K}{\delta+v}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}\equiv\xi_{\max}$,

where $b_{11}\equiv(\delta+v)\beta_Q^{\frac{\beta_Q}{\beta_Q-1}}>0$, $b_{12}\equiv(\delta+v)(\delta+\rho)\beta_Q^{\frac{\beta_Q}{\beta_Q-1}}>0$,

$b_{13}\equiv(\delta+v)\beta_K+(\delta+\rho)(1-\beta_Q)>0$, and $b_{14}\equiv\beta_K(1-\beta_Q)\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}>0$.

Note that the denominator of the right hand side of (A2.1) is always positive for the given range. Since the numerator is a quadratic function of ξ with positive intercept and the denominator converge to positive infinitesimal, we have $\lim_{\xi\to 0+}\phi^k(\xi)=\infty$.

Now we prove $\lim_{\xi\to\xi_{\max}-}\phi^k(\xi)=-\infty$ as follows.

First, rewrite (A2.1) as follows.

$$\phi^k(\xi)=\frac{(\delta+\nu)(\delta+\rho)z^{-1}\xi^2-\{(\delta+\rho)(1-\beta_Q)+(\delta+\nu)\beta_K\}\xi}{\sigma(\beta_K\xi-(\delta+\nu)z^{-1}\xi^2)}+\frac{b_{14}}{\sigma(\beta_K-b_{11}\xi)\xi}$$

$$=\frac{(\delta+\rho)\{(\delta+\nu)z^{-1}\xi^2-\beta_K\xi\}-\{(\delta+\rho)(1-\beta_Q)-(\rho-\nu)\beta_K\}\xi}{\sigma(\beta_K\xi-(\delta+\nu)z^{-1}\xi^2)}+\frac{\beta_K(1-\beta_Q)z}{\sigma(\beta_K-b_{11}\xi)\xi}$$

$$=\frac{\beta_K(1-\beta_Q)z-\{(\delta+\rho)(1-\beta_Q)-(\rho-\nu)\beta_K\}\xi}{\sigma(\beta_K-b_{11}\xi)\xi}-\frac{\delta+\rho}{\sigma}, \qquad \text{(A2.2)}$$

where $z\equiv\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}$.

Let $D(\xi)$ denote the numerator of the left term of (A2.2). The limit of $D(\xi)$ as ξ to $\xi_{\max}$ is as follows.

$$\lim_{\xi\to\xi_{\max}-}D(\xi)=\beta_K(1-\beta_Q)z-\{(\delta+\rho)(1-\beta_Q)-(\rho-\nu)\beta_K\}\frac{\beta_K z}{\delta+\nu}$$

$$=\frac{(\delta+\nu)\beta_K(1-\beta_Q)z-\{(\delta+\rho)(1-\beta_Q)-(\rho-\nu)\beta_K\}\beta_K z}{\delta+\nu}$$

$$=\frac{(\rho-\nu)(\beta_Q-1)\beta_K z+(\rho-\nu)\beta_K{}^2 z}{\delta+\nu}=\frac{(\rho-\nu)(\beta_Q+\beta_K-1)\beta_K z}{\delta+\nu}$$

$$=\frac{-(\rho-\nu)\beta_K\beta_L z}{\delta+\nu}<0$$

Since the corresponding denominator is positive infinitesimal, we have proven $\lim_{\xi\to\xi_{\max}-}\phi^k(\xi)=-\infty$.

Due to the fact that the denominator of (A2.1) is positive and the sign of limit towards upper bound is negative, it is necessary that $\Omega(\xi_{\max})<0$. Since $\Omega(0)=b_{14}>0$, there exists a unique $\bar{\xi}$ such that $\Omega(\bar{\xi})=0$. $\bar{\xi}$ is the smaller root of the equation $\Omega=0$, i.e. $\bar{\xi}\equiv\frac{b_{13}-\sqrt{b_{13}{}^2-4b_{12}b_{14}}}{2b_{12}}=\frac{\beta_K}{\delta+\rho}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}$. It is easy to see $\bar{\xi}<\xi_{\max}$.

Hence, $\Omega(\xi) \gtreqless 0$, if and only if $\xi \lesseqgtr \bar{\xi}$. Because the denominator is positive,

we have proven that $\phi^k(\xi) \gtreqless 0$, if and only if $\xi \lesseqgtr \bar{\xi} \equiv \frac{\beta_K}{\delta+\rho}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}$.

Moreover, we have shown $\lim_{\xi \to 0+} \phi^k(\xi) = \infty$ and $\lim_{\xi \to \xi_{\max}-} \phi^k(\xi) = -\infty$.

Q.E.D.

Appendix A3 Derivation of *A* matrix

Each element of the matrix ***A*** is derived as follows. Note that the derivation often utilises the fact that $\bar{\phi}^{\boldsymbol{G}} = \bar{\phi}_2^{\boldsymbol{p}} = \bar{\phi}^{\boldsymbol{k}} = 0$, in which the overline denotes the steady-state value.

(1) $$\left.\frac{\partial G\phi^G}{\partial G}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = \left.\frac{\partial}{\partial G}\left(pF^W - \delta G\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = \bar{p}\left.\frac{\partial F^W}{\partial G}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} - \delta =$$

$$= \bar{p}\frac{\varepsilon_G\left(\bar{G}\right)F^W\left(\bar{G}\right)}{\bar{G}} - \delta .$$

By substituting $\varepsilon_G\left(\bar{G}\right) = \frac{\delta+\rho}{\delta}$ and $\frac{F^W\left(\bar{G}\right)}{\bar{G}} = \frac{\delta}{\bar{p}}$ into the above expression, we obtain $\left.\frac{\partial G\phi^G}{\partial G}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = \rho$.

(2) $$\left.\frac{\partial G\phi^G}{\partial p}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = \left.\frac{\partial}{\partial p}\left(pF^W - \delta G\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = F^W\left(\bar{G}\right) = \frac{\delta\bar{G}}{\bar{p}} .$$

(3) $$\left.\frac{\partial G\phi^G}{\partial \hat{k}}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = \left.\frac{\partial}{\partial \hat{k}}\left(pF^W - \delta G\right)\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = 0 .$$

(4) $\left.\frac{\partial p\phi_2^P}{\partial G}\right|_{\boldsymbol{x}^e} = \bar{p}\left.\frac{\partial \phi_2^P}{\partial G}\right|_{\boldsymbol{x}^e}$.

$$\left.\frac{\partial \phi_2^P}{\partial G}\right|_{\boldsymbol{x}^e} = \frac{\partial}{\partial G}\left[\frac{F^W\{(1-\varepsilon_G)\delta+\rho\}+\hat{K}\phi_p^k\{(1-\varepsilon_G)\delta+\rho-\varepsilon_G\phi^G\}}{\left(F^W+\hat{K}\phi_p^k\right)H_1} - H_2\phi^k\right]\Bigg|_{\boldsymbol{x}^e}$$

$$= \frac{\partial}{\partial G}\left[\frac{F^W\{(1-\varepsilon_G)\delta+\rho\}+\hat{K}\phi_p^k\{(1-\varepsilon_G)\delta+\rho-\varepsilon_G\phi^G\}}{\left(F^W+\hat{K}\phi_p^k\right)H_1}\right]\Bigg|_{\boldsymbol{x}^e} - \left.\left(\phi^k\frac{\partial H_2}{\partial G}+H_2\frac{\partial \phi^k}{\partial G}\right)\right|_{\boldsymbol{x}^e},$$

where $H_1 \equiv \varepsilon_{pp} - \sigma\varepsilon_p - \frac{\varepsilon_p^{\phi_p^k}\hat{K}\phi_p^k}{F^W+\hat{K}\phi_p^k}$ and

$$H_2 \equiv \frac{\left(F^W+\hat{K}\phi_p^k\right)\left(\varepsilon_{pk}-\sigma\varepsilon_k\right)-\left(1+\varepsilon_k^{\phi_p^k}\right)\hat{K}\phi_p^k}{\left(F^W+\hat{K}\phi_p^k\right)H_1}.$$

Since $\left.\left(\phi^k\frac{\partial H_2}{\partial G}+H_2\frac{\partial \phi^k}{\partial G}\right)\right|_{\boldsymbol{x}^e} = 0$, we have

$$\left.\frac{\partial \phi_2^P}{\partial G}\right|_{\boldsymbol{x}^e} = \frac{\partial}{\partial G}\left[\frac{F^W\{(1-\varepsilon_G)\delta+\rho\}+\hat{K}\phi_p^k\{(1-\varepsilon_G)\delta+\rho-\varepsilon_G\phi^G\}}{\left(F^W+\hat{K}\phi_p^k\right)H_1}\right]\Bigg|_{\boldsymbol{x}^e}$$

$$= \frac{1}{\left(\bar{F}^W+\bar{K}\bar{\phi}_p^k\right)D_1}\frac{\partial}{\partial G}\left[F^W\{(1-\varepsilon_G)\delta+\rho\}+\hat{K}\phi_p^k\{(1-\varepsilon_G)\delta+\rho-\varepsilon_G\phi^G\}\right]\Big|_{\boldsymbol{x}^e}$$

$$-\frac{\bar{F}^W\{(1-\bar{\varepsilon}_G)\delta+\rho\}+\bar{K}\bar{\phi}_p^k\{(1-\bar{\varepsilon}_G)\delta+\rho-\bar{\varepsilon}_G\bar{\phi}^G\}}{\left\{\left(\bar{F}^W+\bar{K}\bar{\phi}_p^k\right)D_1\right\}^2}\frac{\partial}{\partial G}\left\{\left(F^W+\hat{K}\phi_p^k\right)H_1\right\}\Big|_{\boldsymbol{x}^e}$$

$$= \frac{1}{\left(\bar{F}^W+\bar{K}\bar{\phi}_p^k\right)D_1}\frac{\partial}{\partial G}\left[F^W\{(1-\varepsilon_G)\delta+\rho\}-\hat{K}\phi_p^k\left\{\varepsilon_G\left(\delta+\phi^G\right)-\delta-\rho\right\}\right]\Big|_{\boldsymbol{x}^e} - 0$$

$$= \frac{\left[\left\{ \left(1-\bar{\varepsilon}_G\right)\delta + \rho \right\} \frac{dF^W}{dG} - \delta F^W \frac{d\varepsilon_G}{dG} - \hat{K}\phi_p^k \left\{ \left(\delta + \phi^G\right) \frac{d\varepsilon_G}{dG} + \varepsilon_G \frac{\partial \phi^G}{\partial G} \right\} \right]\Bigg|_{\boldsymbol{x}^e}}{\left(\bar{F}^W + \bar{K}\bar{\phi}_p^k\right) D_1}$$

$$= \frac{1}{\left(\bar{F}^W + \bar{K}\bar{\phi}_p^k\right) D_1} \left\{ -\delta\left(\bar{F}^W + \bar{K}\bar{\phi}_p^k\right) \frac{d\varepsilon_G}{dG}\bigg|_{\boldsymbol{x}^e} - \bar{K}\bar{\phi}_p^k \bar{\varepsilon}_G \frac{\partial \phi^G}{\partial G}\bigg|_{\boldsymbol{x}^e} \right\},$$

where $D_1 \equiv \bar{\varepsilon}_{pp} - \sigma\bar{\varepsilon}_p - \left(\frac{\bar{p}\bar{K}}{\bar{F}^W + \bar{K}\bar{\phi}_p^k} \right) \frac{\partial \phi_p^k}{\partial p}\bigg|_{\boldsymbol{x}^e}$.

Here, $\frac{d\varepsilon_G}{dG}\bigg|_{\boldsymbol{x}^e} = \frac{d}{dG}\left(\frac{G}{F^W} \frac{dF^W}{dG} \right)\bigg|_{\boldsymbol{x}^e}$

$$= \frac{1}{\left\{F^W\left(\bar{G}\right)\right\}^2} \left(F^W\left(\bar{G}\right) - \bar{G} \frac{dF^W}{dG}\bigg|_{\bar{G}} \right) \frac{dF^W}{dG}\bigg|_{\bar{G}} + \frac{\bar{G}}{F^W\left(\bar{G}\right)} \frac{d^2 F^W}{dG^2}\bigg|_{\bar{G}}$$

$$= \frac{1}{\bar{G}} \varepsilon_G\left(\bar{G}\right)\left\{1 - \varepsilon_G\left(\bar{G}\right)\right\} + \frac{\bar{p}}{\delta} \frac{d^2 F^W}{dG^2}\bigg|_{\bar{G}} = \frac{1}{\bar{G}} \left(\frac{\delta + \rho}{\delta} \right)\left(-\frac{\rho}{\delta} \right) + \frac{\bar{p}}{\delta} \frac{d^2 F^W}{dG^2}\bigg|_{\bar{G}}, \text{ and}$$

$$\frac{\partial \phi^G}{\partial G}\bigg|_{\boldsymbol{x}^e} = \frac{\partial}{\partial G}\left(\frac{pF^W}{G} - \delta \right)\bigg|_{\boldsymbol{x}^e} = \frac{\bar{p}}{\bar{G}^2}\left(G\frac{dF^W}{dG} - F^W \right)\bigg|_{\boldsymbol{x}^e} = \frac{\bar{p}\bar{F}^W}{\bar{G}^2}\left(\bar{\varepsilon}_G - 1\right) = \frac{\rho}{\bar{G}}.$$

Hence,

$$\frac{\partial p\phi_2^P}{\partial G}\bigg|_{\boldsymbol{x}^e} = \bar{p} \frac{\left(\bar{F}^W + \bar{K}\bar{\phi}_p^k\right)\left\{ \frac{\rho}{\bar{G}}\left(\frac{\delta + \rho}{\delta} \right) - \bar{p} \frac{d^2 F^W}{dG^2}\bigg|_{\bar{G}} \right\} - \bar{K}\bar{\phi}_p^k \bar{\varepsilon}_G \frac{\rho}{\bar{G}}}{\left(\bar{F}^W + \bar{K}\bar{\phi}_p^k\right) D_1}$$

$$= \frac{\bar{p}}{D_1}\left\{ \frac{\rho}{\bar{G}}\left(\frac{\delta + \rho}{\delta} \right)\left(\frac{\bar{F}^W}{\bar{F}^W + \bar{K}\bar{\phi}_p^k} \right) - \bar{p} \frac{d^2 F^W}{dG^2}\bigg|_{G=\bar{G}} \right\}.$$

Let's evaluate $\bar{\phi}_p^k$ and D_1.

$$\bar{\phi}_p^k \equiv \frac{\partial \phi^k}{\partial p}\bigg|_{\boldsymbol{x}^e} = \frac{\partial \phi^k}{\partial \xi}\frac{\partial \xi}{\partial p}\bigg|_{\boldsymbol{x}^e} = b_6 \frac{\bar{\xi}}{\bar{p}} \frac{\partial \phi^k}{\partial \xi}\bigg|_{\boldsymbol{x}^e}, \text{ where}$$

$$\frac{\partial \phi^k}{\partial \xi}\bigg|_{\boldsymbol{x}^e} = \frac{2b_{12}\bar{\xi} - b_{13}}{\sigma\left(\beta_K - b_{11}\bar{\xi}\right)\bar{\xi}} - \phi^k\bigg|_{\boldsymbol{x}^e} \frac{\frac{\partial}{\partial \xi}\left\{\sigma\left(\beta_K - b_{11}\xi\right)\xi\right\}\bigg|_{\boldsymbol{x}^e}}{\sigma\left(\beta_K - b_{11}\bar{\xi}\right)\bar{\xi}}$$

$$= \frac{2b_{12}\overline{\xi} - b_{13}}{\sigma\left(\beta_K - b_{11}\overline{\xi}\right)\overline{\xi}} - 0 = -\frac{(\delta+\rho)\{\rho\beta_K + (\delta+\rho)\beta_L\}}{\rho\sigma\beta_K\overline{\xi}} = -\frac{b_{16}}{\overline{\xi}}, \text{ in which}$$

$$b_{11} \equiv (\delta+\nu)\beta_Q^{\frac{\beta_Q}{\beta_Q - 1}} > 0, \quad b_{12} \equiv (\delta+\nu)(\delta+\rho)\beta_Q^{\frac{\beta_Q}{\beta_Q - 1}} > 0,$$

$$b_{13} \equiv (\delta+\nu)\beta_K + (\delta+\rho)\left(1-\beta_Q\right) > 0, \ \overline{\xi} \equiv \frac{\beta_K}{\delta+\rho}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}, \text{ and}$$

$$b_{16} \equiv \frac{(\delta+\rho)\{\rho\beta_K + (\delta+\rho)\beta_L\}}{\rho\sigma\beta_K} > 0.$$

Hence $\overline{\phi}_p^k = -\dfrac{b_6 b_{16}}{\overline{p}}$.

$$\left.\frac{\partial\phi_p^k}{\partial p}\right|_{\boldsymbol{x}^e} = \left.\frac{\partial}{\partial p}\left[\frac{b_6}{p}\left\{\frac{\left(2b_{12}\xi - b_{13}\right) - \phi^k\sigma\left(\beta_K - 2b_{11}\xi\right)}{\sigma\left(\beta_K - b_{11}\xi\right)}\right\}\right]\right|_{\boldsymbol{x}^e}$$

$$= -\frac{b_6\overline{\xi}}{\overline{p}^2}\left\{\frac{2b_{12}\overline{\xi} - b_{13}}{\sigma\left(\beta_K - b_{11}\overline{\xi}\right)}\right\} + \left.\frac{b_6}{\overline{p}}\frac{\partial\xi}{\partial p}\frac{d}{d\xi}\left\{\frac{\left(2b_{12}\xi - b_{13}\right) - \phi^k\sigma\left(\beta_K - 2b_{11}\xi\right)}{\sigma\left(\beta_K - b_{11}\xi\right)}\right\}\right|_{\boldsymbol{x}^e}$$

$$= \frac{b_6 b_{16}}{\overline{p}^2} + \left.\frac{b_6{}^2\overline{\xi}}{\overline{p}^2}\frac{d}{d\xi}\left\{\frac{\left(2b_{12}\xi - b_{13}\right) - \phi^k\sigma\left(\beta_K - 2b_{11}\xi\right)}{\sigma\left(\beta_K - b_{11}\xi\right)}\right\}\right|_{\boldsymbol{x}^e}$$

$$= \frac{b_6 b_{16}}{\overline{p}^2} + \frac{b_6{}^2\overline{\xi}}{\overline{p}^2}\left\{\frac{2b_{12}}{\sigma\left(\beta_K - b_{11}\overline{\xi}\right)} - \frac{\beta_K - 2b_{11}\overline{\xi}}{\left(\beta_K - b_{11}\overline{\xi}\right)}\left.\frac{\partial\phi^k}{\partial\xi}\right|_{\boldsymbol{x}^e}\right\}$$

$$= \frac{b_6 b_{16}}{\overline{p}^2}\left\{2 - \frac{\delta}{\rho} + \frac{2\delta(\delta+\rho)b_6}{\rho\sigma b_{16}}\right\} = \frac{b_6 b_{16}}{\overline{p}^2}\left[2 - \frac{\delta}{\rho} + \frac{2\delta\beta_K\beta_Q}{\left(1-\beta_Q\right)\{\rho\beta_K + (\delta+\rho)\beta_L\}}\right].$$

Thus,

$$D_1 \equiv \overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p - \left(\frac{b_6 b_{16}\overline{K}}{\overline{p}\overline{F}^W - b_6 b_{16}\overline{K}}\right)\left[2 - \frac{\delta}{\rho} + \frac{2\delta\beta_K\beta_Q}{\left(1-\beta_Q\right)\{\rho\beta_K + (\delta+\rho)\beta_L\}}\right]$$

$$= \overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p + b_{17}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right)^{-1}, \text{ in which}$$

$$b_{17} \equiv 2 - \frac{\delta}{\rho} + \frac{2\delta\beta_K\beta_Q}{\left(1-\beta_Q\right)\{\rho\beta_K + (\delta+\rho)\beta_L\}}.$$

As a result,

$$\left.\frac{\partial p\phi_2^P}{\partial G}\right|_{\boldsymbol{x}^e} = \frac{\overline{p}}{D_1}\left\{\frac{\rho(\delta+\rho)}{\delta\overline{G} - b_6 b_{16}\overline{K}} - \overline{p}\left.\frac{d^2F^W}{dG^2}\right|_{\overline{G}}\right\}$$

(5) $$\left.\frac{\partial p\phi_2^P}{\partial p}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = \left.\phi^p\right|_{\boldsymbol{x}=\boldsymbol{x}^e} + \overline{p}\left.\frac{\partial\phi^p}{\partial p}\right|_{\boldsymbol{x}=\boldsymbol{x}^e} = 0 + \overline{p}\left.\frac{\partial\phi^p}{\partial p}\right|_{\boldsymbol{x}=\boldsymbol{x}^e}.$$

$$\left.\frac{\partial\phi_2^P}{\partial p}\right|_{\boldsymbol{x}^e} = \frac{0 + \left.\left[\hat{K}\left\{(1-\varepsilon_G)\delta + \rho - \varepsilon_G\phi^G\right\}\dfrac{\partial\phi_p^k}{\partial p} + \hat{K}\phi_p^k\dfrac{\partial}{\partial p}\left\{(1-\varepsilon_G)\delta + \rho - \varepsilon_G\phi^G\right\}\right]\right|_{\boldsymbol{x}^e}}{\left(\overline{F}^W + \overline{K}\phi_p^k\right)D_1}$$

$$-\frac{\overline{F}^W\left\{(1-\overline{\varepsilon}_G)\delta + \rho\right\} + \overline{K}\overline{\phi}_p^k\left\{(1-\overline{\varepsilon}_G)\delta + \rho - \overline{\varepsilon}_G\overline{\phi}^G\right\}}{\left(F^W + \hat{K}\phi_p^k\right)^2 H_1^{\,2}}\left.\frac{\partial}{\partial p}\left\{\left(F^W + \hat{K}\phi_p^k\right)H_1\right\}\right|_{\boldsymbol{x}^e}$$

$$-\overline{\phi}^k\left.\frac{\partial H_2}{\partial p}\right|_{\boldsymbol{x}^e} - \left\{\frac{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right) - \left(1 + \overline{\varepsilon}_k^{\phi_p^k}\right)\overline{K}\overline{\phi}_p^k}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1}\right\}\left.\frac{\partial\overline{\phi}^k}{\partial p}\right|_{\boldsymbol{x}^e}$$

$$= \frac{0 - \overline{\varepsilon}_G\overline{K}\overline{\phi}_p^k\left.\dfrac{\partial\phi^G}{\partial p}\right|_{\boldsymbol{x}^e}}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1} - 0 - 0 - \left\{\frac{\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right)}{D_1} - \left(1 + \overline{\varepsilon}_k^{\phi_p^k}\right)\frac{\overline{K}\overline{\phi}_p^k}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1}\right\}\overline{\phi}_p^k$$

$$= -\frac{\overline{K}\overline{\phi}_p^k(\delta+\rho)/\overline{p}}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1} - \left[\frac{\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right)}{D_1} - \frac{\overline{K}}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1}\left\{\overline{\phi}_p^k + \overline{k}\left.\frac{\partial\phi_p^k}{\partial\hat{k}}\right|_{\boldsymbol{x}^e}\right\}\right]\overline{\phi}_p^k$$

$$= -\left[\frac{\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right)}{D_1} - \frac{\overline{K}}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1}\left\{\overline{\phi}_p^k - \frac{\delta+\rho}{\overline{p}} + \overline{k}\left.\frac{\partial\phi_p^k}{\partial\hat{k}}\right|_{\boldsymbol{x}^e}\right\}\right]\overline{\phi}_p^k.$$

$$= \frac{b_6 b_{16}}{\overline{p}}\left[\frac{\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right)}{D_1} - \frac{\overline{K}}{\left(\overline{F}^W + \overline{K}\overline{\phi}_p^k\right)D_1}\left\{-\frac{b_6 b_{16}}{\overline{p}} - \frac{\delta+\rho}{\overline{p}} + \overline{k}\left.\frac{\partial\phi_p^k}{\partial\hat{k}}\right|_{\boldsymbol{x}^e}\right\}\right].$$

Here we need to derive $\left.\dfrac{\partial\phi_p^k}{\partial\hat{k}}\right|_{\boldsymbol{x}^e}$.

$$\left.\frac{\partial\phi_p^k}{\partial\hat{k}}\right|_{\boldsymbol{x}^e} = \left.\frac{\partial}{\partial\hat{k}}\left[\frac{b_6}{p}\left\{\frac{\left(2b_{12}\xi - b_{13}\right) - \phi^k\sigma\left(\beta_K - 2b_{11}\xi\right)}{\sigma\left(\beta_K - b_{11}\xi\right)}\right\}\right]\right|_{\boldsymbol{x}^e}$$

$$= \frac{b_6}{p}\frac{\partial \xi}{\partial \hat{k}}\frac{\partial}{\partial \xi}\left\{\frac{(2b_{12}\xi - b_{13}) - \phi^k \sigma(\beta_K - 2b_{11}\xi)}{\sigma(\beta_K - b_{11}\xi)}\right\}\Bigg|_{\boldsymbol{x}^e}$$

$$= \frac{b_6 b_8 \bar{\xi}}{\sigma \bar{p}\bar{k}}\left[\frac{2b_{12}(\beta_K - b_{11}\xi) + b_{11}(2b_{12}\xi - b_{13}) - (\beta_K - b_{11}\xi)\frac{\partial}{\partial \xi}\{\phi^k \sigma(\beta_K - 2b_{11}\xi)\} + 0}{(\beta_K - b_{11}\xi)^2}\right]\Bigg|_{\boldsymbol{x}^e}$$

$$= \frac{b_6 b_8 \bar{\xi}}{\sigma \bar{p}\bar{k}}\left[\frac{2b_{12}\beta_K - b_{11}b_{13}}{(\beta_K - b_{11}\xi)^2} - \frac{\sigma(\beta_K - 2b_{11}\xi)\frac{\partial \phi^k}{\partial \xi} + 0}{(\beta_K - b_{11}\xi)}\right]\Bigg|_{\boldsymbol{x}^e}$$

$$= \frac{b_6 b_8}{\bar{p}\bar{k}}\left\{b_{16} - \frac{2\delta(\delta+\rho)^2 \beta_L}{\rho^2 \sigma \beta_K}\right\}.$$

Hence, $\left.\frac{\partial p\phi_2^P}{\partial p}\right|_{\boldsymbol{x}^e} = \frac{b_6 b_{16}}{D_1} \times$

$$\left[(\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k) - \frac{\bar{K}}{(\bar{F}^W - \bar{K}\, b_6 b_{16}/\bar{p})}\left\{(b_8 - 1)\frac{b_6 b_{16}}{\bar{p}} - \frac{\delta+\rho}{\bar{p}} - \frac{2b_6 b_8 \delta(\delta+\rho)^2 \beta_L}{\bar{p}\rho^2\sigma\beta_K}\right\}\right]$$

$$= \frac{b_6 b_{16}}{D_1}\left[(\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k) - \left\{b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \frac{\delta+\rho}{b_6 b_{16}}\right\}\left(1 - \frac{\delta\bar{G}}{b_6 b_{16}\bar{K}}\right)^{-1}\right].$$

(6) $\left.\frac{\partial p\phi_2^P}{\partial \hat{k}}\right|_{\boldsymbol{x}^e} = \bar{p}\left.\frac{\partial \phi_2^P}{\partial \hat{k}}\right|_{\boldsymbol{x}^e}.$

$$\left.\frac{\partial \phi_2^P}{\partial \hat{k}}\right|_{\boldsymbol{x}^e} = \frac{\left.\frac{\partial}{\partial \hat{k}}\left[F^W\{(1-\varepsilon_G)\delta + \rho\}\right]\right|_{\boldsymbol{x}^e} + \{(1-\bar{\varepsilon}_G)\delta + \rho - \bar{\varepsilon}_G\bar{\phi}^G\}\left.\frac{\partial}{\partial \hat{k}}\left(\hat{K}\phi_p^k\right)\right|_{\boldsymbol{x}^e}}{(\bar{F}^W + \bar{K}\bar{\phi}_p^k)D_1}$$

$$- \frac{\bar{F}^W\{(1-\bar{\varepsilon}_G)\delta + \rho\} + \{(1-\bar{\varepsilon}_G)\delta + \rho - \bar{\varepsilon}_G\bar{\phi}^G\}}{(\bar{F}^W + \bar{K}\bar{\phi}_p^k)^2 D_1^{\,2}}\left.\frac{\partial}{\partial \hat{k}}\{(F^W + \hat{K}\phi_p^k)H_1\}\right|_{\boldsymbol{x}^e}$$

$$-\frac{\dfrac{\partial}{\partial \hat{k}}\left[\left\{\left(F^{W}+\hat{K}\phi_p^k\right)\left(\varepsilon_{pk}-\sigma\varepsilon_k\right)-\left(1+\varepsilon_k^{\phi_p^k}\right)\hat{K}\phi_p^k\right\}\phi^k\right]\Bigg|_{\boldsymbol{x}^e}}{\left(\overline{F}^{W}+\overline{K}\overline{\phi}_p^k\right)D_1}$$

$$+\frac{\left\{\left(\overline{F}^{W}+\overline{K}\overline{\phi}_p^k\right)\left(\overline{\varepsilon}_{pk}-\sigma\overline{\varepsilon}_k\right)-\left(1+\overline{\varepsilon}_k^{\phi_p^k}\right)\overline{K}\overline{\phi}_p^k\right\}\overline{\phi}^k}{\left(\overline{F}^{W}+\overline{K}\overline{\phi}_p^k\right)^2 D_1^{\,2}}\frac{\partial}{\partial \hat{k}}\left[\left(F^{W}+\hat{K}\phi_p^k\right)H_1\right]\Bigg|_{\boldsymbol{x}^e}$$

$$=0-0-\frac{\dfrac{\partial}{\partial \hat{k}}\left[\left\{\left(F^{W}+\hat{K}\phi_p^k\right)\left(\varepsilon_{pk}-\sigma\varepsilon_k\right)-\left(1+\varepsilon_k^{\phi_p^k}\right)\hat{K}\phi_p^k\right\}\phi^k\right]\Bigg|_{\boldsymbol{x}^e}}{\left(\overline{F}^{W}+\overline{K}\overline{\phi}_p^k\right)D_1}+0$$

$$=0-\frac{\left\{\left(\overline{F}^{W}+\overline{K}\overline{\phi}_p^k\right)\left(\overline{\varepsilon}_{pk}-\sigma\overline{\varepsilon}_k\right)-\overline{K}\left(\overline{\phi}_p^k+\overline{k}\,\dfrac{\partial\phi_p^k}{\partial\hat{k}}\bigg|_{\boldsymbol{x}^e}\right)\right\}\dfrac{\partial\phi^k}{\partial\hat{k}}\bigg|_{\boldsymbol{x}^e}}{\left(\overline{F}^{W}+\overline{K}\overline{\phi}_p^k\right)D_1},$$

in which $\dfrac{\partial\phi^k}{\partial\hat{k}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=\dfrac{\partial\phi^k}{\partial\xi}\dfrac{\partial\xi}{\partial\hat{k}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=b_8\,\dfrac{\overline{\xi}}{\overline{k}}\dfrac{\partial\phi^k}{\partial\xi}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}$,

where $b_8\equiv\dfrac{\beta_L}{1-\beta_Q}$.

Hence,

$$\frac{\partial p\phi_2^P}{\partial\hat{k}}\bigg|_{\boldsymbol{x}^e}=\overline{p}\,\frac{\partial\phi_2^P}{\partial\hat{k}}\bigg|_{\boldsymbol{x}^e}=$$

$$=\frac{b_8 b_{16}\overline{p}}{D_1\overline{k}}\left[\left(\overline{\varepsilon}_{pk}-\sigma\overline{\varepsilon}_k\right)-\left\{b_7+\frac{2b_8\delta\left(\delta+\rho\right)^2\beta_L}{b_{16}\rho^2\sigma\beta_K}\right\}\left(1-\frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right)^{-1}\right].$$

(7) $$\frac{\partial\hat{k}\phi^k}{\partial G}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=\overline{k}\,\frac{\partial\phi^k}{\partial G}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=\overline{k}\,\frac{\partial}{\partial G}\left\{\frac{b_{12}\xi^2-b_{13}\xi+b_{14}}{\sigma(\beta_K-b_{11}\xi)\xi}\right\}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=0.$$

(8) $$\frac{\partial\hat{k}\phi^k}{\partial p}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=\overline{k}\,\frac{\partial\phi^k}{\partial p}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=\overline{k}\,\frac{\partial\phi^k}{\partial\xi}\frac{\partial\xi}{\partial p}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}=\overline{k}b_6\,\frac{\overline{\xi}}{\overline{p}}\frac{\partial\phi^k}{\partial\xi}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}$$

$$= \bar{k} b_6 \frac{\bar{\xi}}{\bar{p}} \left[- \frac{(\delta+\rho)\{\rho\beta_K + (\delta+\rho)\beta_L\}}{\rho\sigma\beta_K \bar{\xi}} \right] = -b_6 b_{16} \frac{\bar{k}}{\bar{p}} .$$

(9) $$\frac{\partial \hat{k}\phi^k}{\partial \hat{k}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e} = \phi^k \bigg|_{\boldsymbol{x}=\boldsymbol{x}^e} + \bar{k} \frac{\partial \phi^k}{\partial \hat{k}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e} = 0 + \bar{k} \frac{\partial \phi^k}{\partial \hat{k}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e}$$

$$= \bar{k} \frac{\partial \phi^k}{\partial \xi} \frac{\partial \xi}{\partial \hat{k}}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e} = \bar{k} b_8 \frac{\bar{\xi}}{\bar{k}} \frac{\partial \phi^k}{\partial \xi}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e} = -b_8 b_{16} .$$

As a result, the Jacobian matrix evaluated at the steady state is given as

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \text{ where } A_{11} \equiv \rho > 0 , \; A_{12} \equiv \frac{\delta \bar{G}}{\bar{p}} > 0 ,$$

$$A_{21} \equiv \frac{\bar{p}}{D_1} \left\{ \frac{\rho(\delta+\rho)}{\delta\bar{G} - b_6 b_{16} \bar{K}} - \bar{p} \frac{d^2 F^W}{dG^2}\bigg|_{\bar{G}} \right\} ,$$

$$A_{22} \equiv \frac{b_6 b_{16}}{D_1} \left[\left(\bar{\varepsilon}_{pk} - \sigma \bar{\varepsilon}_k \right) - \left\{ b_7 + \frac{2 b_8 \delta (\delta+\rho)^2 \beta_L}{b_{16} \rho^2 \sigma \beta_K} + \frac{\delta+\rho}{b_6 b_{16}} \right\} \left(1 - \frac{\delta \bar{G}}{b_6 b_{16} \bar{K}} \right)^{-1} \right] ,$$

$$A_{23} \equiv \frac{b_8 b_{16} \bar{p}}{D_1 \bar{k}} \left[\left(\bar{\varepsilon}_{pk} - \sigma \bar{\varepsilon}_k \right) - \left\{ b_7 + \frac{2 b_8 \delta (\delta+\rho)^2 \beta_L}{b_{16} \rho^2 \sigma \beta_K} \right\} \left(1 - \frac{\delta \bar{G}}{b_6 b_{16} \bar{K}} \right)^{-1} \right] ,$$

$$A_{32} \equiv -b_6 b_{16} \frac{\bar{k}}{\bar{p}} < 0 , \; A_{33} \equiv -b_8 b_{16} < 0 , \text{ in which}$$

$$D_1 \equiv \bar{\varepsilon}_{pp} - \sigma \bar{\varepsilon}_p + b_{17} \left(1 - \frac{\delta \bar{G}}{b_6 b_{16} \bar{K}} \right)^{-1} .$$

Moreover, it is necessary to evaluate $\bar{\varepsilon}_{pp} - \sigma \bar{\varepsilon}_p$ and $\bar{\varepsilon}_{pk} - \sigma \bar{\varepsilon}_k$ for analysing the local stability condition. They are evaluated as follows.

Recall Eq. (3.9) with $\xi \equiv p^{b_6} \hat{k}^{b_8}$.

$$\hat{c} = \hat{k} \frac{(b_{10} - b_9 \xi)}{b_1 p^{1-\varphi} \xi} \left(1 + \frac{\beta}{\beta_1 - b_{11} \xi} \right), \text{ where } b_9 \equiv \delta + \nu - \frac{\delta+\rho}{\sigma} ,$$

$$b_{10} \equiv \frac{(\sigma-1)\beta_1\beta_2^{\frac{\beta_2}{1-\beta_2}}}{\sigma} > 0\text{, and } b_{11} \equiv (\delta+\nu)\beta_2^{\frac{\beta_2}{\beta_2-1}} > 0\,.$$

From this expression, the price elasticity of consumption ε_p is evaluated as

$$\varepsilon_p \equiv \frac{\partial \hat{c}}{\partial p}\frac{p}{\hat{c}}$$

$$= \varphi - 1 + \frac{b_6\xi\{2b_9b_{11}\xi - b_9(1-\beta_Q) - b_{10}b_{11}\}}{(b_9\xi - b_{10})\{b_{11}\xi - (1-\beta_Q)\}} - \frac{b_6(\beta_K - 2b_{11}\xi)}{\beta_K - b_{11}\xi}, \tag{A3.1}$$

and the elasticity of consumption with respect to private capital stock ε_k is

$$\varepsilon_k \equiv \frac{\partial \hat{c}}{\partial \hat{k}}\frac{\hat{k}}{\hat{c}} = 1 + \frac{b_8\xi\{2b_9b_{11}\xi - b_9(1-\beta_Q) - b_{10}b_{11}\}}{(b_9\xi - b_{10})\{b_{11}\xi - (1-\beta_Q)\}} - \frac{b_8(\beta_K - 2b_{11}\xi)}{\beta_K - b_{11}\xi}. \tag{A3.2}$$

Further, we can derive

$$\varepsilon_{pp} \equiv \frac{\partial c_p}{\partial p}\frac{p}{c_p} = \varepsilon_p - 1 + b_6\frac{\xi}{\varepsilon_p}\frac{\partial \varepsilon_p}{\partial \xi}\text{, and} \tag{A3.3}$$

$$\varepsilon_{pk} \equiv \frac{\partial c_p}{\partial \hat{k}}\frac{\hat{k}}{c_p} = \varepsilon_k + b_8\frac{\xi}{\varepsilon_p}\frac{\partial \varepsilon_p}{\partial \xi}. \tag{A3.4}$$

The far right partial derivative term is evaluated at

$$\frac{\partial \varepsilon_p}{\partial \xi} = b_6\left[-\frac{b_9b_{10}}{(b_9\xi - b_{10})^2} - \frac{(1-\beta_Q)b_{11}}{\{b_{11}\xi - (1-\beta_Q)\}^2} + \frac{\beta_K b_{11}}{(\beta_K - b_{11}\xi)^2}\right]. \tag{A3.5}$$

From equations (A3.1) - (A3.4) we obtain

$$\frac{\varepsilon_{pk} - \sigma\varepsilon_k}{\varepsilon_{pp} - \sigma\varepsilon_p} = \frac{\beta_L}{\beta_Q} + \frac{\beta_L\{1-(1-\phi)(\sigma-1)\} - \beta_Q(\sigma-1)}{\beta_Q(\varepsilon_{pp} - \sigma\varepsilon_p)}. \tag{A3.6}$$

From (A3.1), (A3.3) and (A3.5) with $\bar{\xi} = \frac{\beta_K}{\delta+\rho}\beta_Q^{\frac{\beta_Q}{1-\beta_Q}}$,

$$\bar{\varepsilon}_{pp} - \sigma\bar{\varepsilon}_p = \bar{\varepsilon}_p - 1 + b_6\frac{\bar{\xi}}{\bar{\varepsilon}_p}\frac{\partial \varepsilon_p}{\partial \xi}\bigg|_{\boldsymbol{x}=\boldsymbol{x}^e} - \sigma\bar{\varepsilon}_p$$

$$= (1-\sigma)\bar{\varepsilon}_p - 1$$

$$+ \frac{(\delta+\rho)\beta_Q{}^2}{\bar{\varepsilon}_p(1-\beta_Q)^2}\left[\frac{\delta}{\rho^2} - \frac{(\sigma-1)\{(\sigma-1)\delta-\rho\}}{\rho^2\sigma^2} - \frac{\delta(1-\beta_Q)\beta_K}{\{\rho\beta_K+(\delta+\rho)\beta_L\}^2}\right],$$

in which $\bar{\varepsilon}_p = \varphi - 1 + \frac{(\delta+\rho)\beta_Q}{\rho(1-\beta_Q)}\left\{\frac{1}{\sigma} - \frac{\rho(1-\beta_Q)}{\rho\beta_K+(\delta+\rho)\beta_L}\right\}.$

$\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k$ is easily obtained by putting the above into Eq. (A3.6) as

$$\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k = \frac{\beta_L}{\beta_Q}\left(\bar{\varepsilon}_{pp} - \sigma\bar{\varepsilon}_p\right) + \frac{\beta_L\{1-(1-\phi)(\sigma-1)\} - \beta_Q(\sigma-1)}{\beta_Q}.$$

Note that $\bar{\varepsilon}_{pp} - \sigma\bar{\varepsilon}_p$ and $\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k$ are determined by parameters only.

Appendix A4 Proof of Proposition 9

Let $g(r)$ denote the characteristic polynomial of $\boldsymbol{A}$. We have

$$g(r) \equiv -r^3 + \text{trace}\,\boldsymbol{A} \times r^2 + \left(A_{12}A_{21} - A_{11}A_{22} - A_{11}A_{33}\right)r + \det \boldsymbol{A}, \qquad \text{(A4.1)}$$

where $\text{trace}\,\boldsymbol{A} \equiv A_{11} + A_{22} + A_{33}$ and

$$\det \boldsymbol{A} \equiv A_{11}\left(A_{22}A_{33} - A_{23}A_{32}\right) - A_{12}A_{21}A_{33}.$$

The eigenvalues of $\boldsymbol{A}$ are the roots of the characteristic equation $g(r) = 0$.

First, we have $g(0) = \det \boldsymbol{A}$. $\det \boldsymbol{A}$ is evaluated as

$$\det \boldsymbol{A} = \rho\left(A_{22}A_{33} - A_{23}A_{32}\right) - \frac{\delta\overline{G}}{\overline{p}}A_{21}A_{33}$$

$$= \rho\frac{b_6b_8b_{16}^2}{D_1}\left(\frac{\delta+\rho}{b_6b_{16}}\right)\left(1 - \frac{\delta\overline{G}}{b_6b_{16}\overline{K}}\right)^{-1} + \frac{\delta\overline{G}}{D_1}\left\{\frac{\rho(\delta+\rho)}{\delta\overline{G} - b_6b_{16}\overline{K}} - \overline{p}\frac{d^2F^W}{dG^2}\bigg|_{\overline{G}}\right\}b_8b_{16}$$

$$= \frac{b_8b_{16}\rho(\delta+\rho)}{D_1}\left\{\left(\frac{b_6b_{16}\overline{K}}{b_6b_{16}\overline{K} - \delta\overline{G}}\right) - \left(\frac{\delta\overline{G}}{b_6b_{16}\overline{K} - \delta\overline{G}}\right)\right\} - b_8b_{16}\frac{\delta\overline{p}\overline{G}}{D_1}\frac{d^2F^W}{dG^2}\bigg|_{\overline{G}}$$

$$= \frac{b_8 b_{16}}{D_1}\left\{\rho(\delta+\rho) - \delta\overline{p}\overline{G}\,\frac{d^2F^W}{dG^2}\bigg|_{\overline{G}}\right\}.$$

Because of diminishing marginal water product ($\frac{d^2F^W}{dG^2}$ < 0), we have

$$\det \boldsymbol{A} \;\gtrless\; 0 \text{ if and only if } D_1 \;\gtrless\; 0.$$

Recall $D_1 \equiv \overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p + b_{17}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right)^{-1}$. Baseline parameter values result in b_{17} = 1.343 > 0. With the assumption that $\delta\overline{G} < b_6 b_{16}\overline{K}$ or equivalently $\overline{K} > 0.793\overline{G}$ with baseline parameter values, which seems universally to hold in the real world, we have the inequality $D_1 > \overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p + b_{17} = 0.103 > 0$. As a result baseline parameter values establish det $\boldsymbol{A}$ > 0. Similarly, the same assumption with the same parameter values establish trace $\boldsymbol{A}$ < 0 as proven below.

First, $A_{22} < \dfrac{b_6 b_{16}\left[\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right) - \left\{b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \frac{\delta+\rho}{b_6 b_{16}}\right\}\right]}{\overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p + b_{17}}$ holds as follows.

$$A_{22} = \frac{b_6 b_{16}\left[\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right) - \left\{b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \frac{\delta+\rho}{b_6 b_{16}}\right\}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right)^{-1}\right]}{\overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p + b_{17}\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right)^{-1}}$$

$$= \frac{b_6 b_{16}\left[\left(\overline{\varepsilon}_{pk} - \sigma\overline{\varepsilon}_k\right)\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right) - \left\{b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \frac{\delta+\rho}{b_6 b_{16}}\right\}\right]}{\left(\overline{\varepsilon}_{pp} - \sigma\overline{\varepsilon}_p\right)\left(1 - \frac{\delta\overline{G}}{b_6 b_{16}\overline{K}}\right) + b_{17}}$$

$< \dfrac{-b_6 b_{16}\left\{b_7 + \dfrac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \dfrac{\delta+\rho}{b_6 b_{16}}\right\}}{b_{17}}$, of which the far right inequality holds because 0< $\bar{\varepsilon}_{pp} - \sigma\bar{\varepsilon}_p + b_{17}$ < $(\bar{\varepsilon}_{pp} - \sigma\bar{\varepsilon}_p)\eta + b_{17}$ < b_{17} and $\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k$ < $(\bar{\varepsilon}_{pk} - \sigma\bar{\varepsilon}_k)\eta < 0$ in which $\eta \equiv 1 - \dfrac{\delta\bar{G}}{b_6 b_{16}\bar{K}} \in (0,1)$ < 1 by the assumption.

Hence, trace $\boldsymbol{A} = A_{11} + A_{22} + A_{33}$

$$< A_{11} - \frac{b_6 b_{16}}{b_{17}}\left\{b_7 + \frac{2b_8\delta(\delta+\rho)^2\beta_L}{b_{16}\rho^2\sigma\beta_K} + \frac{\delta+\rho}{b_6 b_{16}}\right\} + A_{33} = -\,0.356 < 0.$$

Due to the fact that $g(r) = 0$ is a cubic function with negative parameter of cubic term with $g(0) = \det \boldsymbol{A} > 0$, there must be at least one positive real root. Let α denote this positive real root ($\alpha > 0$). We can rewrite $g(r)$ as

$$g(r) = -(r-\alpha)(r^2 + k_1 r + k_2) = -r^3 + (\alpha - k_1)r^2 + (\alpha k_1 - k_2)r + \alpha k_2 . \quad \text{(A4.2)}$$

It is easy to see that the other two eigenvalues are given as $r = \dfrac{-k_1 \pm \sqrt{k_1^2 - 4k_2}}{2}$.

By comparing Eq. (A4.1) with Eq. (A4.2), we have

$$k_1 = \alpha - \text{trace}\,\boldsymbol{A} > 0 \quad \text{and} \quad k_2 = \frac{\det \boldsymbol{A}}{\alpha} > 0 .$$

As a result, there are two eigenvalues which are either negative real numbers or two complex conjugate numbers with negative real parts, in addition to the positive real eigenvalue α.

Recall that the linearised system around the optimal steady state is in general unstable unless as many initial conditions as the number of unstable eigenvalues are freely chosen (Gandolfo 1997; Theorem 18.3). Because the system has one control variable, i.e. the rate of water charge $p(t)$, while there is only one unstable eigenvalues, it is possible to stabilise the optimal trajectories around the optimal steady-state by choosing proper initial rate of water charge $p(0)$.

Q.E.D.

Appendix A5 List of Parameters

ν : rate of population growth

σ : elasticity of marginal felicity

φ : weight of market good consumption in satisfaction production

δ : depreciation rate

ρ : rate of pure time preference

β_K : factor share of private capital in market commodity production

β_Q : factor share of water in market commodity production

β_L : factor share of labour in market commodity production

$$b_1 \equiv \varphi^{-\varphi}(1-\varphi)^{-(1-\varphi)},\ b_2 \equiv \frac{1-\varphi}{\sigma},\ b_3 \equiv \frac{(\sigma-1)(1-\varphi)}{\sigma},\ b_4 \equiv \nu - \frac{\rho}{\sigma},\ b_5 \equiv \frac{\sigma-1}{\sigma},$$

$$b_6 \equiv \frac{\beta_Q}{1-\beta_Q},\ b_7 \equiv \frac{\beta_K}{1-\beta_Q},\ b_8 \equiv \frac{\beta_L}{1-\beta_Q},\ b_9 \equiv \delta + \nu - \frac{\delta+\rho}{\sigma},$$

$$b_{10} \equiv \frac{(\sigma-1)\beta_K \beta_Q^{\frac{\beta_Q}{1-\beta_Q}}}{\sigma},\ b_{11} \equiv (\delta+\nu)\beta_Q^{\frac{\beta_Q}{\beta_Q-1}},\ b_{12} \equiv (\delta+\nu)(\delta+\rho)\beta_Q^{\frac{\beta_Q}{\beta_Q-1}},$$

$$b_{13} \equiv (\delta+\nu)\beta_K + (\delta+\rho)(1-\beta_Q),\ b_{14} \equiv \beta_K(1-\beta_Q)\beta_Q^{\frac{\beta_Q}{1-\beta_Q}},$$

$$b_{15} \equiv \varphi^{\varphi}(1-\varphi)^{1-\varphi}(\delta+\rho)^{\frac{\beta_Q-1}{\beta_L}}\beta_K^{\frac{\beta_K}{\beta_L}}\beta_Q^{\frac{\beta_Q}{\beta_L}}\{(\rho-\nu)\beta_K + (\delta+\rho)\beta_L\},$$

$$b_{16} \equiv \frac{(\delta+\rho)\{\rho\beta_K + (\delta+\rho)\beta_L\}}{\rho\sigma\beta_K},\ \text{and}\ b_{17} \equiv 2 - \frac{\delta}{\rho} + \frac{2\delta\beta_K\beta_Q}{(1-\beta_Q)\{\rho\beta_K + (\delta+\rho)\beta_L\}}.$$