Should environmental goods be discounted hyperbolically ?

- A general perspective on good–specific discount rates -

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Abstract: I analyze social discount rates in the presence of limited substitutability in welfare between different forms of consumption. In a generalized DU framework I derive good–specific social rates of discount as generators of marginal utility development. Applied to the case of produced goods and environmental amenities it is shown how non–coinciding growth rates lead to hyperbolic discounting. Looking at the evaluation of a small project I give an interpretation of the good–specific (social) rates of discount and clarify how they have to be applied. The notation used highlights how in general magnitude and form of discounting depend on different economically plausible perspectives and the choice of numeraire.

Keywords: discounting, hyperbolic, limited substitutability, numeraire dependence, social rate of discount, environmental discount rate

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1 Introduction

Introduced by Samuelson (1937) and provided with an axiomatic foundation by Koopmans (1960) the discount utility (DU) model became the predominant framework for intertemporal decision analysis. In recent years more and more of its standard specifications have been challenged into question (for a survey consult Frederick et al. 2002). One of the disconcertments bears on the generally applied exponential discount function. Experiments indicate that observed behavior is better described by the use of hyperbolic (i.e. falling) discount rates (Frederick et al. 2002:378). Other promoters for hyperbolic discount rates are models based on reasoning on intergenerational justice such as Chichilnisky (1996) and Li and Löfgren (2000).

One problem of hyperbolic discounting is that it can lead to continuous revision of the (formerly) optimal plan, a phenomenon called time inconsistency. One way to "solve" this problem is to look at the planning process as a non–cooperative game against one's future selves or future generations (Phelps and Pollak 1968, Arrow 1999). Another access to this problem is set forth by Weitzman (1998) and Azfar (1999) who rationalize hyperbolic discounting in the case of uncertainty. More recently and in a context closer to my own Gollier (2002) gave sufficient conditions for hyperbolic discounting in an uncertain world of economic growth.

Last year hyperbolic discounting also obtained official status in the political process of evaluation when the new British Green Book prescribed hyperbolic discount rates for the evaluation of long–term projects (HM Treasury 2003:97 et sqq.).

In his paper "On the 'Environmental' Discount Rate" Weitzman (1994) presents a reasoning how the consideration of environmental amenities being degraded or being luxury goods can lead to hyperbolic discounting. He sees it as a rigorous foundation for the demand of environmentalists to apply lower and hyperbolic discount rates in long–term cost–benefit calculations. The interpretation of the effect derived by Weitzman as a discount rate is criticized by Arrow et al. (1995:140). This paper interrelates to the controversy tackling the question of an environmental discount rate from a slightly different perspective in a quite different representation.

My model will yield a hyperbolic discount rate based on the assumption of limited substitutability between produced and environmental goods within a time consistent framework. I develop a general notation that allows to switch easily between different perspectives on discounting. Unlike Weitzman I explicitly model the two–good case and explain why Weitzman's 'environmental' discount rate is indeed contestable. I show that none the less a meaningful lower and hyperbolic discount rate for the environment can follow from limited substitutability and that the rejection of Arrow et al. does not apply in general.

Following this introduction section 2 introduces good–specific discount factors that equal time propagators of marginal utility generated by good–specific discount rates. It is shown that for the widely used one–commodity DU model such a generator of marginal utility development coincides with the social discount rate. I summarize the interpretation of the terms making up the latter.

Section 3 returns to the two–commodity case and focuses on appearing substitutability effects. The reasoning is discussed in application to the case of an aggregate of environmental amenities and another aggregate of produced consumption commodities. Assuming that the growth rate of consumption is higher for produced goods than for environmental goods, I obtain hyperbolic discount rates for both commodities with the good–specific rate for the environmental amenities being lower.

In section 4 I introduce prices into the analysis by looking at the optimal control problem of a representative consumer. Price evolvement over time is seen to be the product of marginal utility propagation and (inverse) capital value propagation. It is observed that, if the two goods are not complete substitutes, there is no canonical discount rate and capital productivity and good–specific discount rates generally cease to coincide.

Section 5 looks at the evaluation of a small project in a notation that allows to switch easily between different perspectives on discounting. It is worked out how good–specific discount rates have to be treated in the process of evaluation and how the choice of numeraire affects the discount rate. The critique of Arrow et al. (1995:140) on the environmental discount rate is taken up and reviewed and social discounting is related to market based evaluation. Section 6 concludes.

2 Marginal utility and the rate of discount

This section analyzes how marginal utility develops over time and how this relates to the rate of discount. I study the case of two consumption goods, but the model is easily extended to the case of N goods. The consumption quantities of the two goods are labeled $x_1(t)$ and $x_2(t)$. To simplify notation the time argument will generally be omitted. With $\mathsf{x}_i : [0,T] \to \mathbb{R}$ I denote the consumption path for good i from time $t = 0$ to $t = T$, where $T = \infty$ is allowed. x comprises the consumption paths of the two goods in vector notation. Welfare is taken to be that of a representative consumer.[1] It is assumed to be of the form

$$(1) \qquad \mathcal{U} = \int_0^T U(x_1(t), x_2(t), t)\, dt \ .$$

The function $U(x_1(t), x_2(t), t)$ will be called the (instantaneous) utility function. It is required to be twice differentiable. The general part of the paper (on good–specific discount rates and their interpretation) does not assume a specific time dependence. For a given consumption path x I write $U^{\mathsf{x}}(t) \equiv U(x_1(t), x_2(t), t)$. Similar definitions apply to the derivatives of U. Again for notational simplification the x will usually be dropped. I define the *time propagator of marginal utility* $D_i^{\mathsf{x}}(t, t_0)$ for a given consumption path x by

$$(2) \qquad D_i^{\mathsf{x}}(t, t_0) \equiv \frac{\frac{\partial U(x_1,x_2,t)}{\partial x_i}(t)}{\frac{\partial U(x_1,x_2,t)}{\partial x_i}(t_0)}$$

$$\Leftrightarrow \frac{\partial U(x_1, x_2, t)}{\partial x_i}(t) \equiv D_i^{\mathsf{x}}(t, t_0) \frac{\partial U(x_1, x_2, t)}{\partial x_i}(t_0) \quad , i \in \{1, 2\} \ .$$

The time propagator $D_i^{\mathsf{x}}(t, t_0)$ captures the time development of marginal utility by relating the value of marginal utility between any two points of time t and t_0 in a multiplicative form. In discrete time Malinvaud (1974:234) calls the $D_i^{\mathsf{x}}(t, t_0)$ discount factors, a wording I will adopt in my continuous time setting. As I look at a representative consumer I will also call $D_i^{\mathsf{x}}(t, t_0)$ a *social discount factor*. In what follows it will become clear that $D_i^{\mathsf{x}}(t, t_0)$ generally does not coincide with the factor multiplied to a time independent instantaneous utility function –

[1] A reader who refuses the meaningfulness of a representative consumer can regard $\mathcal{U}$ as being a given social welfare function that is only formally equivalent to individual utility reasoning. Only in section 4 and the last equation of section 5 where price paths are introduced this picture makes the analysis a bit more complicated.

representing pure time preference – which is also called a discount factor.

A closer look at the time dependence of the propagator brings about its corresponding discount rate. For this purpose let me write out the infinitesimal time propagator of marginal utility as follows:

$$D_i^{\mathsf{x}}(t+dt,t) = \frac{\frac{\partial U}{\partial x_i}(t+dt)}{\frac{\partial U}{\partial x_i}(t)} = 1 + \frac{\frac{\partial U}{\partial x_i}(t+dt) - \frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t)}$$
$$= 1 + \underbrace{\frac{\frac{\partial^2 U}{\partial t \partial x_i}(t) + \frac{\partial^2 U}{\partial x_i^2}(t)\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t)\dot{x}_j}{\frac{\partial U}{\partial x_i}(t)}}_{\equiv -\delta_i(x(t),\dot{x}(t),t) \equiv -\delta_i(t)} dt \qquad {\scriptstyle i,j\in\{1,2\}\text{ with } i\neq j.}$$

The instantaneous[2] change of D_i^{x} is completely characterized by $\delta_i(t) \equiv \delta_i(x(t),\ \dot{x}(t),t) \equiv -[\frac{\partial^2 U}{\partial t \partial x_i}(t) + \frac{\partial^2 U}{\partial x_i^2}(t)\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t)\dot{x}_j]/\frac{\partial U}{\partial x_i}(t)$ which corresponds to a discount rate (see below). In mechanics (the negative of) $\delta_i(t)$ is called the generator of D_i^{x} as it describes, or – from an active point of view – generates, the change of D_i^{x}.[3] In the context of this paper $\delta_i(t)$ can be understood as being the (good–specific) *generator of time development of marginal utility.* I find it helpful to keep this picture in mind when talking about time development and discounting.

The finite time propagator follows from the infinitesimal one as derived in Appendix A:

$$(3) \qquad D_i^{\mathsf{x}}(t,t_0) = e^{-\int_{t_0}^{t} \delta_i(x(t'),\dot{x}(t'),t')dt'} = e^{\int_{t_0}^{t} \frac{\frac{\partial^2 U}{\partial t' \partial x_i}(t') + \frac{\partial^2 U}{\partial x_i^2}(t')\dot{x}_i + \frac{\partial^2 U}{\partial x_j \partial x_i}(t')\dot{x}_j}{\frac{\partial U}{\partial x_i}(t')} dt'} .$$

Thus the finite time propagators are completely determined by the families $\delta_i(t')_{,t'\in[0,T]}$, $i \in \{1,2\}$ (requiring given consumption path x). By checking the relation $\delta(t) = -\frac{\dot{D}(t)}{D(t)}$ (e.g. Laibson 1997:449) with t_0 constant, it can be verified that δ_i is indeed the instantaneous discount rate corresponding to $D_i^{\mathsf{x}}(t,t_0)$. Recapitulating, the discount rates $\delta_i(t)$ define or generate the instantaneous changes of D_i^{x} which on their part comprise the intertemporal development of welfare. For the rest of this section I will only be interested in the instantaneous discount rates

[2]Here by instantaneous is meant that the first and the second argument of D_i^{x} differ only infinitesimally.

[3]Compare Sakurai (1985:46 et sqq.,71 et sq.) or Goldstein (1980:chapter 9) for this view on mechanics (e.g. momentum being the generator of translation). The minus sign is introduced to meet the economic perspective of positively discounting instead of negatively "upcounting".

$\delta_i(t)$. I will come back to the propagators $D_i^{\mathsf{X}}(t, t_0)$ in sections 4 and 5.

In models with a single (aggregate) consumption good $\delta_i(t)$ is known as the (instantaneous) social rate of time preference or *social discount rate*. This stands out more clearly if instantaneous utility is specified to the form usually applied in discount utility models: $U(x_1, x_2, t) = u(x_1, x_2)e^{-\rho t}$. Let me neglect the second commodity for the moment by setting it constant.[4] Then the discount rate $\delta \equiv \delta_1$ becomes

$$(4) \qquad \delta(t) = \rho - \frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}} \dot{x}_1 = \rho - \frac{\partial \frac{\partial u}{\partial x_1}}{\partial x_1} \frac{x_1}{\frac{\partial u}{\partial x_1}} \frac{\dot{x}_1}{x_1} = \rho + \theta_{(x(t))} \; \hat{x}_{1(x_1(t), \dot{x}_1(t))} \; .$$

This expression for the social discount rate is well known in the literature. For a detailed discussion of the terms I refer to Arrow et al.(1995:136) or Pearce et al.(2003:130). The constant ρ is called the pure rate of time preference. θ is the (absolute[5] of the) elasticity of marginal utility of consumption. $\hat{x}_1$ denotes the growth rate of the consumption commodity.

In many macroeconomic models u is assumed to exhibit constant elasticity of intertemporal substitution (CIES). This leads to constancy of the term $\theta \, \hat{x}_1$ and thereby to a constant social discount rate δ in the steady state.[6] A constant rate of discount goes along with exponential discounting, i.e. a discount factor of the form $D_i^{\mathsf{X}}(t, t_0) = e^{-\delta t}$ (compare equation 3).

In general the terms in equation (4) don't have to be constant. In fact the term $\theta \, \hat{x}_1$ is also used to argue for hyperbolic discounting. A discount function is said to be hyperbolic if it is characterized by a falling instantaneous discount rate (Laibson 1997:450). Confronted with the question of discounting the long–term impacts of global warming, it is argued that global warming can result in a decline of consumption growth and thus diminish the discount rate that should be applied to the far future. Translating equation (4) to a framework with uncertainty Gollier (2002) works out further conditions that lead to a falling discount rate by the term $\theta \, \hat{x}_1$.

[4] x_2 can be regarded as a fixed parameter of the utility function.

[5] As in the standard DU models diminishing but positive marginal utility in consumption is assumed $-\frac{\partial^2 u}{\partial x_1^2} / \frac{\partial u}{\partial x_1} = \theta$ turns out to be positive.

[6] The elasticity of intertemporal substitution is the inverse of the elasticity of marginal utility θ. Thus in the steady state $\hat{x}_1$ and θ are both constant (Barro and Sala-i-Martin 1995:64).

3 Two goods with limited substitutability

Coming back to the model with two consumption goods and $U(x_1, x_2, t) = u(x_1, x_2)e^{-\rho t}$ equation (4) has to be modified. The discount rate corresponding to the propagator of marginal utility $D_1^{\mathsf{X}}(t, t_0)$ becomes

$$(5) \qquad \delta_1(t) = \rho - \frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}}\dot{x}_1 - \frac{\frac{\partial^2 u}{\partial x_1 \partial x_2}}{\frac{\partial u}{\partial x_1}}\dot{x}_2 \; .$$

It comprises an additional term that depends on the substitutability[7] $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ between the two goods. Working out its nature I take instantaneous utility to be of the functional form $u(x_1, x_2) = [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{1/s}$, $s \in \mathbb{R}, a_1, a_2 \in \mathbb{R}_+$.[8] This furthers understanding as it separates good–specific utility $u_i(x_i)$ from substitutability effects parameterized in a simple form by s. As derived in appendix B the corresponding discount rate turns out to be

$$(6) \quad \delta_1(t) = \rho - \frac{\frac{\partial^2 u_1}{\partial x_1^2}}{\frac{\partial u_1}{\partial x_1}}\dot{x}_1 - (1-s)\frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s}\left(\frac{\frac{\partial u_2}{\partial x_2}(x_2)}{u_2(x_2)}\dot{x}_2 - \frac{\frac{\partial u_1}{\partial x_1}(x_1)}{u_1(x_1)}\dot{x}_1\right).$$

The first and the second term of equation (6) resemble the widely used equation (4). In what follows I want to examine the additional third term that depends on the substitutability parameter s. Having recognized this, I simplify the utility function by setting $u(x_1) = x_1$ and $u(x_2) = x_2$ which leads to the standard CES utility function $u(x_1, x_2) = [a_1 x_1^s + a_2 x_2^s]^{1/s}$.[9] Thereby I eliminate in equation (6) the well studied second term and simplify the third without changing its dependence on the substitutability parameter s. This step leads to the discount

[7]See Coto-Millán (1999:21) for different ways of defining substitutability of consumption goods.

[8]For $s = 0$ the a_i are restricted to $a_1 + a_2 = 1$ and the functional form is defined by the limit $s \to 0$ leading to $u(x_1, x_2) = u_1(x_1)^{a_1} u_2(x_2)^{a_2}$. For $s \to -\infty, \infty$ the limit functions are found to be $\min\{u_1(x_1), u_2(x_2)\}$ and $\max\{u_1(x_1), u_2(x_2)\}$ respectively.

[9]CES functions exhibit constant elasticity of substitution σ that relates to s by the formula $\sigma = \frac{1}{1-s}$. For its derivation see Arrow et al. (1961). Observe that CES functions are homogeneous of degree one. Thus proportional overall growth does not change marginal utility as it follows that the latter is homogeneous of degree zero in consumption. This explains why the chosen functional form is so well suited to focus on the new effect due to relative difference in growth, filtering out the overall growth effect extensively discussed in the literature in connection with equation (4).

rate:

$$\delta_1(t) = \rho - \underbrace{(1-s)\frac{a_2 x_2^s}{a_1 x_1^s + a_2 x_2^s}}_{\equiv G_s(x_1,x_2)} (\hat{x}_2 - \hat{x}_1)\,. \tag{7}$$

Interpreting this expression I want to specify the two goods. Let the first characterize an aggregate of environmental goods[10] and let the second denote an aggregate of produced goods. It makes the model more viable not to think of x_1 as the mere quantity of environmental consumption, but to think of it as some kind of quality weighted measure of consumption of environmental amenities.[11] The same interpretation should hold for the measurement x_2 of the produced good. I *assume* that the rate of growth in (quality adjusted) consumption of the produced good $\hat{x}_2$ is *higher* than that of the environmental good $\hat{x}_1$. This difference can be due to ecological reasons limiting the growth (and amount) of environmental consumption stronger than for technically produced consumption goods or to the problem of the environment being an undersupplied public good. I have the first interpretation in mind.

It is instructive to look first at the two special cases where $s = 1$ and $s = 0$. The first corresponds to the additive utility function $u(x_1, x_2) = a_1 x_1 + a_2 x_2$ and leads to $G_1(x_1, x_2) = 0$. Thus in the case of completely substitutable goods the *additional term vanishes* and equations (4) and (6) coincide. No further insight is gained by explicitly modeling two goods.

For $s = 0$ and $a_1 + a_2 = 1$ the utility function takes the Cobb–Douglas form $u(x_1, x_2) = x_1^{a_1} + x_2^{a_2}$ (Arrow et al. 1961:231) and it is found that $G_0(x_1, x_2) = a_2$. Hence the discount rate becomes

$$\delta_1(t) = \rho - a_2 (\hat{x}_2 - \hat{x}_1)\,. \tag{8}$$

Equation (8) states that the discount rate of the environmental good is reduced

[10] Environmental goods can be defined by being "generally consumed on site, with little or no transformation by ordinary productive processes" (Fisher and Krutilla 1975:360).

[11] Quantity and quality in consumption of environmental goods can develop at different rates. This makes the simplification of just looking at some aggregate a strong assumption as will become clear during my analysis of the consequences of differing growth rates for environmental and produced goods. Though for the sake of comprehensibility it is advisable to focus only on one such effect and commit to the simplification above.

by a term proportional to the difference in the growth rates of the two goods. In a steady state the terms in equation (8) are constant and hence *discounting of the environmental good stays exponential with a lower discount rate* for Cobb–Douglas utility.

Now I turn to the more general case of limited substitutability characterized by $0 < s < 1$. I call this parameter range limited substitutability because utility can be gained by consuming only one of the goods, but still mixtures are preferred. G_s is positive and hence the additional term in equation (7) again reduces the discount rate. But this time G_s is constantly decreasing which can be seen by the following transformation derived in appendix C:

$$(9) \qquad G_s = (1-s)\frac{a_2 x_2(t)^s}{a_1 x_1(t)^s + a_2 x_2(t)^s} = (1-s)\frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t') - \hat{x}_1(t')\,dt'} + 1} .$$

From $\hat{x}_2 - \hat{x}_1 > 0$ it follows that the expression $\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s\int_0^t \hat{x}_2(t') - \hat{x}_1(t')\,dt'}$ is monotonously falling to zero. Hence the second factor of G_s grows to one[12] and G_s monotonously grows to $(1-s)$. In a steady state thus the discount rate δ_1 falls monotonously to $\delta_1 = \rho - (1-s)(\hat{x}_2 - \hat{x}_1)$ for $t \to \infty$ (assuming $T = \infty$). Therewith *discounting of the environmental good becomes hyperbolic.* The *discount rate* for the environmental good is *lowered* by the additional term depending on the substitutability between x_1 and x_2. The less substitutable both goods are (i.e. the smaller s is), the lower becomes the discount rate for the environmental good.

Now I turn to the discount rate of the produced good x_2. It is easily arrived at by switching the indices in equation (7):

$$(10) \qquad \delta_2(t) = \rho + \underbrace{(1-s)\frac{a_1 x_1^s}{a_2 x_2^s + a_1 x_1^s}}_{\equiv H_s(x_1, x_2)} (\hat{x}_2 - \hat{x}_1) ,$$

where the plus sign in front of the second term is due to interchanging the positions of $\hat{x}_2$ and $\hat{x}_1$. H_s like G_s is positive and similar to the latter it can be transformed to:

$$H_s = (1-s)\frac{1}{\frac{a_2 x_2(0)^s}{a_1 x_1(0)^s} e^{s\int_0^t \hat{x}_2(t') - \hat{x}_1(t')\,dt'} + 1} .$$

Again this can be seen immediately by switching the indices in equation (9). For

[12] More precisely it must be assumed that there exists $\epsilon > 0$ such that $\hat{x}_1(t) < \hat{x}_2(t) - \epsilon$ for all t to assure that the limit reaches one.

Cobb–Douglas utility ($s = 0$, $a_1 + a_2 = 1$) the *discount rate* is again constant in the steady state, but this time *higher* by $a_1(\hat{x}_2 - \hat{x}_1)$. In the region of limited substitutability ($0 < s < 1$) the term $e^{s\int_0^t \hat{x}_2(t') - \hat{x}_1(t')\,dt'}$ grows monotonously to infinity and thereby H_s falls to zero. This signifies that in a steady state *also the produced good is discounted hyperbolically* with a discount rate falling monotonously to the pure rate of time preference ρ. The interesting result is that, if the growth rates of consumption of two goods differ, an exponentially discounted CES–utility–function in the range of limited substitutability leads to hyperbolic discounting when either of the two goods is considered individually. Observe that for the environmental good x_1 the discount rate will eventually *grow negative* if $(1 - s)(\hat{x}_2(t) - \hat{x}_1(t)) > \rho$, that is, if the difference in growth of consumption between the two goods weighted with the limitedness of substitutability dominates the time preference ρ.[13] The meaning of these individual discount rates will be further discussed in section 5 after introducing prices into the model in section 4.

4 Connecting marginal utility and prices

Up to now I only studied the objective function of the welfare optimization problem and treated the optimal consumption path x as given. To introduce prices into my considerations I take a closer look at the decision problem of the representative consumer. This section introduces his restrictions on consumption by considering his budget constraint. Again the equations of motion for x_1 and x_2 won't be modeled explicitly. I use the budget constraint only to introduce prices which prove helpful in section 5 when the interpretation of discount rates will be discussed. Welfare is again assumed to be of the general form of equation (1) though restricted by the assumptions that follow below. For this section I will assume that the social optimum can be decentralized by an appropriate price system. Prices are measured in units of capital which can be regarded either as money or as real capital. They are denoted by $p_1(t)$ and $p_2(t)$. The interest rate

[13]Observe that this relation determines only the instantaneous discount rate. It is also possible that the discount factor, i.e. the finite time propagator $D_i^{\mathsf{x}}(t, t_0)$, grows bigger than 1, but this is up to the special case.

on capital is $r(t)$. Remuneration for a fixed offer of labor $w(t)$ is only introduced for "completeness" of the budget constraint, but will not play any explicit roll in my further considerations. All these variables are exogenous to the representative consumer. His choice is between saving $\dot{k}(t)$ units of the capital good k and consuming the amounts $x_1(t)$ and $x_2(t)$. Therewith his budget constraint is given by

$$\dot{k}(t) = r(t)k(t) + w(t) - p_1(t)x_1(t) - p_2(t)x_2(t) \ .$$

Thus the Hamiltonian of the representative consumer's optimization problem is

$$H = U(x_1(t), x_2(t), t) + \lambda(t)[r(t)k(t) + w(t) - p_1(t)x_1(t) - p_2(t)x_2(t)] \ .$$

For what follows I shall assume that a sufficiency condition for the optimization problem is met[14] and denote the solution for the consumption path by x. Along this path the following necessary conditions for an optimum must be satisfied:

$$\frac{\partial H}{\partial x_1} = \frac{\partial U}{\partial x_1} - \lambda(t)p_1(t) \stackrel{!}{=} 0 \,, \tag{11}$$

$$\frac{\partial H}{\partial x_2} = \frac{\partial U}{\partial x_2} - \lambda(t)p_2(t) \stackrel{!}{=} 0 \,, \tag{12}$$

$$\frac{\partial H}{\partial k} = \lambda(t)r(t) \stackrel{!}{=} -\dot{\lambda}(t) \ . \tag{13}$$

From equations (11) and (12) I obtain the relations:

$$\frac{\frac{\partial U}{\partial x_1}(t)}{\frac{\partial U}{\partial x_2}(t)} = \frac{p_1(t)}{p_2(t)} \qquad \text{and}$$

$$\frac{\frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t_0)} = \frac{\lambda(t)}{\lambda(t_0)} \frac{p_i(t)}{p_i(t_0)} \qquad i \in \{1, 2\} \ . \tag{14}$$

Integration of equation (13) yields the shadow price of capital

$$\lambda(t) = ce^{-\int_0^t r(t)dt} \quad \text{with the integration constant } \lambda(0) = c \in \mathbb{R}_+ \ .$$

Let me define the *time propagator of capital value* $R(t, t_0) = \frac{\lambda(t)}{\lambda(t_0)} = e^{-\int_{t_0}^t r(t')dt'}$. It relates the shadow price of capital at different points of time.[15] In analogy to the derivation of D_i^{x} in section 2 the (negative of the) productivity of capital $r(t)$ forms the *generator of capital value propagation*. Inserting $R(t, t_0)$ into

[14]See Takayama (1994:660 sqq.) and Chiang (1992:214 et sqq.) for different sufficiency conditions. In addition I assume a continuos control (consumption) path and an interior solution.

[15]For the interpretation of a shadow price (costate variable) compare e.g. Kamien and Schwartz (2000:136 et sqq.).

equation (14) the following relation between the time propagator of marginal utility $D_i^{\mathsf{x}}(t,t_0)$ of good i, the capital value propagator and the price of good i is obtained:

$$(15) \quad p_i(t) = D_i^{\mathsf{x}}(t,t_0)\, p_i(t_0)\, R(t_0,t)\,.$$

Equation (15) shows that time development of (capital measured) prices depends on two influencing factors. One is the effect discussed in sections 2 and 3 depending on the change of marginal utility expressed by $D_i^{\mathsf{x}}(t,t_0)$. The other is generated by the productivity of capital.

Note that it is not the capital value propagator $R(t,t_0)$ that appears in equation (15) but its inverse $R(t_0,t) = R(t,t_0)^{-1} = e^{\int_{t_0}^{t} r(t')dt'}$. This is because prices are measured in units of the capital good. Thus the capital value propagator $R(t,t_0)$ applies to the denominator. Further observe that (15) is a straight forward generalization of the two period relation $(1+\delta)^{-1} \equiv \frac{\partial U}{\partial x}(1)/\frac{\partial U}{\partial x}(0) \stackrel{!}{=} (1+r)^{-1}\frac{p(1)}{p(0)}$ to continuous time.[16]

It is important to note that in the optimum, capital productivity $r(t)$ does generally not equal $\delta_i(t)$. This differs from the single good case the way it is usually put forward. In the latter it can be argued for measuring capital in units of the (only) consumption good. This permits to normalize the consumption good and the capital good at the same time.[17] Such a simplification is not feasible anymore if several consumption goods are considered. This can be seen by deriving the Euler equation from (13) and (11) – respectively (13) and (12) for the second good – and solving for $r(t)$. Yet it can be recognized much easier by the fact that if $r(t) = \delta_i(t)$ for both goods, the different δ_i would have to coincide. But in section 3 I showed that this is usually not the case (compare equations 7 and 10).

[16]Be aware that δ depends on $\mathsf{x} = (x(0), x(1))$.

[17]Compare e.g. Barro and Sala-i-Martin (1995:62). Note that the important trick is that the normalization allows to hold the price ratio of capital and consumption good constant over time, not that prices themselves are actually kept constant. The latter is also done, but this part of the normalization is echoed in the shadow price of the capital good.

5 Discounting and project evaluation

Now I will relate the discount rates derived in sections 2 and 3 to the task of evaluation. To this end I consider changes in consumption and write out the corresponding alterations in welfare using the finite time propagators $D_i^{\mathsf{x}}(t,t_0)$ and $R(t,t_0)$ generated by the rates $\delta_i(t)$ and $r(t)$. I consider a small project that changes the formerly optimal consumption path x^0 to the new path $\mathsf{x}{=}\mathsf{x}^0 + \Delta\mathsf{x}$ with $x_i = x_i^0(t){+}\Delta x_i(t)$, $i \in \{1,2\}$, $t \in [0,T]$. At each point of time $\Delta x_i(t)$ should be small as compared to $x_i(t)$ so that I can develop $U(x_1(t) + \Delta x_1(t), x_2(t) + \Delta x_2(t), t)$ in the $\Delta x_i(t)$ (small project assumption). Hence the welfare of the new consumption path can be written as

$$\begin{aligned}
\mathcal{U} &= \int_0^T U(x_1^0(t) + \Delta x_1(t), x_2^0(t) + \Delta x_2(t), t)\, dt \\
&= \int_0^T U(x_1^0(t), x_2^0(t), t) + \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t) + O(\Delta x(t)^2)\, dt \\
&= \mathcal{U}^0 + \int_0^T \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t) + O(\Delta x(t)^2)\, dt\ , \qquad (16)
\end{aligned}$$

where the marginal utilities are evaluated along x^0. Equation (16) states that neglecting terms of second order in Δx, the project raises welfare if and only if

$$\int_0^T \frac{\partial U}{\partial x_1}(t)\Delta x_1(t) + \frac{\partial U}{\partial x_2}(t)\Delta x_2(t)\, dt > 0\ . \qquad (17)$$

The integral can be considered a cost benefit functional in continuous time with valuation derived from a social welfare optimization. Time specific marginal utilities are used to evaluate the changes in x_1 and x_2 at every point of time. By relating the marginal utilities in equation (17) at different points of time with the help of equation (2) one is taken to the picture of social discounting (equations 18 to 19' below). Before I do this let me choose the reference period $t_0 \equiv 0$ to be the present and assume that in the present there exist prices $p_1(t_0)$ and $p_2(t_0)$ fulfilling $\frac{p_1(t_0)}{p_2(t_0)} = \frac{\frac{\partial U}{\partial x_1}(t_0)}{\frac{\partial U}{\partial x_2}(t_0)}$.[18] These could either be market prices or prices derived from direct methods (e.g. contingent valuation) or indirect methods (e.g. hedonic

[18]For social evaluation of the project I do not assume that the assumptions of section 4 hold, especially I do not presume a given price path. The results of section 4 will only be used when looking at market based evaluation in equation (20).

price studies) of evaluation (compare Hanley et al. 1997:383 et sqq.). Together with equation (2) this relation brings the evaluation functional (17) to the form:

$$(18)\quad \int_0^T D_1^{\mathbf{x}^0}(t,t_0)p_1(t_0)\Delta x_1(t) + D_2^{\mathbf{x}^0}(t,t_0)p_2(t_0)\Delta x_2(t)\,dt > 0\ .$$

By factoring out $D_1^{\mathbf{x}^0}(t,t_0)$ or $D_2^{\mathbf{x}^0}(t,t_0)$ this is equivalent to

$$(19)\quad \int_0^T \left[p_1(t_0)\Delta x_1(t) + \frac{D_2^{\mathbf{x}^0}(t,t_0)}{D_1^{\mathbf{x}^0}(t,t_0)}p_2(t_0)\Delta x_2(t)\right] D_1^{\mathbf{x}^0}(t,t_0)\,dt > 0 \qquad \text{and}$$

$$(19')\quad \int_0^T \left[\frac{D_1^{\mathbf{x}^0}(t,t_0)}{D_2^{\mathbf{x}^0}(t,t_0)}p_1(t_0)\Delta x_1(t) + p_2(t_0)\Delta x_2(t)\right] D_2^{\mathbf{x}^0}(t,t_0)\,dt > 0\ .$$

Equation (18) takes the present prices to determine the relative value of x_1 and x_2 at time $t_0 = 0$ and then propagates both prices using the change of marginal utility over time for the respective good.[19] Another interpretation is to take the $D_i^{\mathbf{x}}(t,t_0)$ as *good–specific discount factors*. Applying this view to the example of section 3 with limited substitutability between produced and environmental goods it would ask to take a *higher and falling discount rate for the aggregate produced good* and a *lower and falling discount rate for the environmental good*. It is important to be aware that *either* one can argue that prices of the environmental good rise due to its increasing relative scarcity, *or* one can apply the individual discount rates discussed above. Doing both at the same time yields a wrong evaluation.

An interesting special case is the evaluation of a project that affects only consumption of the environmental good at different points of time ($\Delta\mathrm{x}_2 = 0$). Then (18) is equivalent to

$$(18')\quad \int_0^T D_1^{\mathbf{x}^0}(t,t_0)\Delta x_1(t)\,dt > 0\,.$$

Hence looking at a *partial model of the environmental sector* optimal discounting can be *hyperbolic and time consistent* at the same time. Time consistency follows, because looking at both sectors, instantaneous utility is discounted exponentially. But the change in optimal consumption of the produced good over time changes marginal utility derived from the environmental good in such a way that in the

[19] Note that these prices $[D_i^{\mathbf{x}}(t,t_0)\,p_i(t_0)]$ are not the same as the capital measured market prices in section 4 (compare equations 15 and 20). $D_i^{\mathbf{x}}(t,t_0)\,p_i(t_0)$ could be called a social price as it determines whether a change in consumption flow raises welfare.

partial model of the environmental sector (only) hyperbolic discount functions lead to time consistent planning.

Equations (19) and (19') take the more usual view, that there is one common discount rate applicable to all goods (*the* discount rate). Equation (19) can be interpreted the following way. Again evaluation starts out with the prices in $t_0 = 0$. Then the first good is taken to be the numeraire (in the sense of keeping its price constant). Hence the change of marginal utility of the first good expressed by $D_1^{\mathbf{\chi}^0}(t, t_0)$ becomes *the* discount factor and the (contemporaneous) value of the second good must be propagated by the relative change of marginal utility of good two relative to good one, i.e. by $\frac{D_2^{\mathbf{\chi}^0}(t,t_0)}{D_1^{\mathbf{\chi}^0}(t,t_0)}$. Applying this perspective again to the example of section 3 the social rate of discount for the environmental good would be *the* discount rate and discounting would take place with the lower hyperbolic discount rate δ_1.

Equation (19') is the analogue taking x_2 to be the numeraire. Related to the case of a produced[20] consumption good and an environmental good this is the perspective that Arrow et al. (1995) take[21] and which is probably prevailing in cost–benefit analysis. Despite the fact that they stress being aware that "relative prices over time (discount factors) will differ from those associated with the consumption good measure" (Arrow et al. 1995:135) when a different numeraire is taken, they criticize environmental economists who ask for a lower or hyperbolic discount rate of environmental goods (Arrow et al. 1995:140). They argue that value propagation of environmental goods is only a question of price propagation or of properly turning environmental amenities into equivalents of produced consumption flow and that this "does not change the discount rate to apply to the consumption stream"(Arrow et al. 1995:140).

I want to make two comments on this. First it is only a matter of perspective which numeraire is taken and if environmental economists dealing primarily with environmental consumption goods prefer to take the perspective of equation (18')

[20]I suppose that Arrow et al. mean produced consumption goods when they discuss discounting of consumption goods versus environmental goods (compare also footnote 10).

[21]Compare Arrow et al. (1995:139):"The appropriate procedure entails converting the environmental change into equivalent contemporaneous [produced] consumption benefits and discounting those."

or (19), then the lower and hyperbolic rate δ_1 becomes *the* discount rate. More interestingly however is the fact that, even in the picture of taking produced consumption to be the numeraire, the limitedness of substitutability between environmental and produced goods does change the discount rate δ_2 to be applied to the consumption stream and turns it hyperbolic (equation 10). This is an effect that Arrow et al. (1995:140) explicitly deny in response to the model of Weitzman (1994).

In fact derivation and interpretation of Weitzman's environmental discount rate are a little cumbersome as he uses a one–sector model and omits to model the environmental good explicitly. The latter is brought into the model only in form of expenditures on environmental amenities that reduce the consumable amount of the produced good. Now Weitzman assumes that the ratio of produced goods spent on the environment grows in income.[22] This way the "consumable part" of additionally produced commodities decreases with growth over time. Weitzman deducts the part of produced consumption that is not consumable from a constant prior discount rate. That way he obtains a new, lower and – by assumption on the functional form of expenditure growth – falling discount rate that he calls an 'environmental' discount rate.

The difficulty with this discount rate is on the one hand that subtraction of environmental expenditure from the prior discount rate happens in a rather ad hoc way and on the other hand that it is not clear what this overall discount rate finally means.

Now section 3 explicitly models the effects that were touched in Weitzman (1994) and in fact can lead to a modified discount rate.[23] My presentation should

[22]Weitzman offers two different intuitions for this assumption. One is that with growing production also environmental degradation rises and the expenditures needed to keep a constant level of environmental amenities grow. The alternative interpretation he offers is that environmental amenities are luxury goods and therefore expenditure for them rises with growing (production based) income.

[23]Weitzman's first interpretation of expenditure growth – i.e. rising costs for keeping a constant level of environmental amenities with growing production (compare footnote 22) – implies the assumption that there is a rise of (marginal) valuation of environmental amenities relative to produced goods. Thus in the sense of section 3 there is limited substitutability and higher growth of produced consumption. This in fact leads to hyperbolic discount rates (though taking

clarify where these effects come from – i.e. shift in (marginal) valuation – and how they have to be interpreted and applied in the case of project evaluation. The substitutability effect derived in section 3 causes the discount rate of the numeraire to depend directly on changes of other forms of consumption and can turn the discount rate hyperbolic. This is an effect that is not captured by mere conversion of the other consumption forms into numeraire equivalents as claimed by Arrow et al. (1995:140).

Before I turn to market based project evaluation I want to point out the main advantage of the perspective of social discounting (equations 18 to 19') as compared to evaluation functional (17). While in (17) marginal utilities have to be evaluated – or in real life estimated – at every point of time the perspective of social discounting bundles this problem in two tasks. First the relative value in the present is estimated and then a functional form for the change of value over time – depending on growth and substitutability of goods – is to be retrieved. Using social discount rates – the generators of the discount factors prescribing marginal utility development – is a perspective especially well suited for the evaluation or estimation of value change over time as economists usually prefer to think in rates and elasticities rather than in absolute values and as it translates multiplicative effects into additive ones. Therefore I perceive a discussion on social discount rates a perspective comparatively well suited for a (highly normative) discourse on societies value shift in the future.

Finally I want to consider the case where markets are complete in the sense that all externalities are reflected in market prices and future markets for all relevant goods exist. This is the case considered in section 4. Applying equation (15) to equation (18) the following evaluation functional for the project is obtained:

$$\int_0^T p_1(t)R(t,t_0)\Delta x_1(t) + p_2(t)R(t,t_0)\Delta x_2(t)\, dt > 0$$

produced consumption as numeraire rather with a higher and not with a lower discount rate!). The luxury good effect offered by Weitzman as an alternative interpretation for growing environmental expenditures can be seen as a combination of limited substitutability with income dependent "expenditure weights" a_1 and a_2 and could thus be modeled similar to my "simple" substitutability effect in section 3.

$$(20) \quad \Leftrightarrow \int_0^T \left[p_1(t)\Delta x_1(t) + p_2(t)\Delta x_2(t)\right] R(t, t_0)\, dt > 0 \; .$$

This time the social discount factor $D_i^{\mathbf{x}}(t, t_0)$ is not needed for evaluation. The price development accounts already for the change in welfare. But as prices are measured in capital and therefore are also influenced by capital productivity. The capital value propagator $R(t, t_0)$ has to correct for this (compare equation 15).[24] Hence capital productivity can be regarded as *the* common discount rate for both goods.

6 Conclusion

To evaluate long–term projects not only relative valuation of different consumption flows in the present is needed but also its development over time. The latter can be represented by social discount factors that describe the time development of marginal utility. In the case of working future markets these social discount factors are reflected in the price paths. Taking these prices to evaluate a small project, the discount rate to be applied is capital productivity.

However, having to decide in the absence of future price paths, which is the case for environmental goods, the marginal utility development itself has to be used to evaluate the project. As the corresponding social discount factors are good–specific one can either use individual social discount rates or choose a numeraire. In the latter case other consumption streams have to be converted into contemporaneous equivalents of the numeraire. A canonical discount rate does not exist. Not only the magnitude but also the form of discounting depends on the choice of the numeraire and its substitutability to other goods.

In the case that environmental and produced goods are of limited substitutability and grow at different rates, either of them acquires a hyperbolic discount rate. The slower growing environmental good yields a lower discount rate. This insight is also important for partial analysis. If a partial model only considers utility

[24] Again it is actually the inverse of the capital value propagator that is applied in the denominator. It is helpful to look at equation (20) the way it appears if the capital price would not be normalized: $\int_0^T \frac{p_1(t)}{R(t_0,t)p_k(t)} \Delta x_1(t) + \frac{p_2(t)}{R(t_0,t)p_k(t)} \Delta x_2(t)\, dt > 0$. In this form it stands out more clearly, that the capital value propagator actually acts on the numeraire p_k.

derived from environmental goods at different points of time then hyperbolic discounting might yield the only time consistent plan.

I hope that my presentation and its interpretation in terms of propagators that can be related to discounting *or* to price development and to individual goods *or* to a numeraire proves helpful in other contexts as well.

Appendix

A Derivation of the finite time propagator of marginal utility

Using the mutltiplicative structure of the propagator the derivation of $D_i^{\mathsf{x}}(t,t_0)$ from $D_i^{\mathsf{x}}(t+dt,t)$ is straightforward:

$$\begin{aligned}
D_i^{\mathsf{x}}(t+dt,t_0) &= \frac{\frac{\partial U}{\partial x_i}(t+dt)}{\frac{\partial U}{\partial x_i}(t_0)} = \frac{\frac{\partial U}{\partial x_i}(t+dt)}{\frac{\partial U}{\partial x_i}(t)}\,\frac{\frac{\partial U}{\partial x_i}(t)}{\frac{\partial U}{\partial x_i}(t_0)} \\
&= D_i^{\mathsf{x}}(t+dt,t)D_i^{\mathsf{x}}(t,t_0) \\
\Rightarrow D_i^{\mathsf{x}}(t+dt,t_0) - D_i^{\mathsf{x}}(t,t_0) &= \underbrace{\left(D_i^{\mathsf{x}}(t+dt,t)-1\right)}_{\parallel}\; D_i^{\mathsf{x}}(t,t_0) \\
&= \overbrace{-\delta_i(x(t),\dot{x}(t),t)\,dt}\; D_i^{\mathsf{x}}(t,t_0) \\
\Rightarrow \frac{D_i^{\mathsf{x}}(t+dt,t_0) - D_i^{\mathsf{x}}(t,t_0)}{dt} &= -\delta_i(x(t),\dot{x}(t),t)\, D_i^{\mathsf{x}}(t,t_0) \\
\Rightarrow \frac{d}{dt} D_i^{\mathsf{x}}(t,t_0) &= -\delta_i(x(t),\dot{x}(t),t)\, D_i^{\mathsf{x}}(t,t_0) \\
\Rightarrow \frac{d}{dt}\ln D_i^{\mathsf{x}}(t,t_0) &= -\delta_i(x(t),\dot{x}(t),t) \\
\Rightarrow D_i^{\mathsf{x}}(t,t_0) &= a\, e^{\int_{t_0}^{t} -\delta_i(x(t'),\dot{x}(t'),t')\,dt'} \,.
\end{aligned}$$

Because of $D_i^{\mathsf{x}}(t,t)=1$ the integration constant a must be equal to 1.

B Calculation of the discount rate for $U(x_1, x_2, t) = [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{1/s} e^{-\rho t}$

The derivatives needed for the computation of δ_1 are for $s \neq 0, 1$:

$$\begin{aligned}
\frac{\partial U}{\partial x_1} &= a_1 u_1(x_1)^{s-1} u_1'(x_1)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1} e^{-\rho t} \ , \\
\frac{\partial^2 U}{\partial x_1^2} &= \left(a_1 u_1(x_1)^{s-1} u_1''(x_1) - (1-s) a_1 u_1(x_1)^{s-2} u_1'(x_1)^2\right)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1} \\
&\quad \cdot e^{-\rho t} + (1-s)(a_1 u_1(x_1)^{s-1})^2 u_1'(x_1)^2 [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2} e^{-\rho t} \quad \text{and} \\
\frac{\partial^2 U}{\partial x_1 \partial x_2} &= (1-s)\left(a_1 u_1(x_1) a_2 u_2(x_2)\right)^{s-1} u_1'(x_1) u_2'(x_2)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2} e^{-\rho t} \ .
\end{aligned}$$

Inserting these into equation (5) yields:

$$\begin{aligned}
\delta_1(t) &= \rho - \frac{\left(a_1 u_1(x_1)^{s-1} u_1''(x_1) - (1-s) a_1 u_1(x_1)^{s-2} u_1'(x_1)^2\right)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}}{a_1 u_1(x_1)^{s-1} u_1'(x_1)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \\
&\quad \cdot \dot{x}_1 - \frac{(1-s)(a_1 u_1(x_1)^{s-1})^2 u_1'(x_1)^2 [a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2}}{a_1 u_1(x_1)^{s-1} u_1'(x_1)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \dot{x}_1 \\
&\quad - \frac{(1-s)\left(a_1 u_1(x_1) a_2 u_2(x_2)\right)^{s-1} u_1'(x_1) u_2'(x_2)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-2}}{a_1 u_1(x_1)^{s-1} u_1'(x_1)[a_1 u_1(x_1)^s + a_2 u_2(x_2)^s]^{\frac{1}{s}-1}} \dot{x}_2 \\
&= \rho - \frac{u_1''(x_1)}{u_1'(x_1)} \dot{x}_1 + (1-s) u_1(x_1)^{-1} u_1'(x_1) \, \dot{x}_1 \\
&\quad -(1-s) \frac{a_1 u_1(x_1)^{s-1} u_1'(x_1)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \dot{x}_1 - (1-s) \frac{a_2 u_2(x_2)^{s-1} u_2'(x_2)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \dot{x}_2 \\
&= \rho - \frac{u_1''(x_1)}{u_1'(x_1)} \dot{x}_1 \\
&\quad +(1-s) \frac{u_1(x_1)^{-1} u_1'(x_1)(a_1 u_1(x_1)^s + a_2 u_2(x_2)^s) - a_1 u_1(x_1)^{s-1} u_1'(x_1)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \dot{x}_1 \\
&\quad -(1-s) \frac{a_2 u_2(x_2)^{s-1} u_2'(x_2)}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \dot{x}_2 \\
&= \rho - \frac{u_1''(x_1)}{u_1'(x_1)} \dot{x}_1 + (1-s) \frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \frac{u_1'(x_1)}{u_1(x_1)} \dot{x}_1 \\
&\quad -(1-s) \frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \frac{u_2'(x_2)}{u_2(x_2)} \dot{x}_2 \ .
\end{aligned}$$

Which brings about equation (6):

$$\delta_1(t) = \rho - \frac{u_1''(x_1)}{u_1'(x_1)} \dot{x}_1 - (1-s) \frac{a_2 u_2(x_2)^s}{a_1 u_1(x_1)^s + a_2 u_2(x_2)^s} \left(\frac{u_2'(x_2)}{a_2 u_2(x_2)} \dot{x}_2 - \frac{u_1'(x_1)}{a_1 u_1(x_1)} \dot{x}_1 \right) \ .$$

Using the functional form given in footnote 8 for $s = 0$ it is an easy calculation to show that this relation also holds for $s = 0$ and $s = 1$.

C Transformation of $\mathbf{G}_s$

First recognize that the following relation holds (similar to that derived in appendix A):

$$\begin{aligned} \frac{d \ln x_i(t)}{dt} &= \frac{\dot{x}_i(t)}{x_i(t)} \\ \Rightarrow d \ln x_i(t)\, dt &= \hat{x}_i(t)\, dt \\ \Rightarrow \quad \ln x_i(t) &= \int_0^t \hat{x}_i(t')\, dt' + c \\ \Rightarrow \quad x_i(t) &= x_i(0) e^{\int_0^t \hat{x}_i(t')\, dt'} \\ \Rightarrow \quad x_i(t)^s &= x_i(0)^s e^{s \int_0^t \hat{x}_i(t')\, dt'} \ . \end{aligned}$$

Therewith $G_s/(1-s)$ can be transformed the following way:

$$\begin{aligned} \frac{G_s}{1-s} &= \frac{a_2 x_2(t)^s}{a_1 x_1(t)^s + a_2 x_2(t)^s} &&= \frac{a_2 x_2(0)^s e^{s \int_0^t \hat{x}_2(t')\, dt'}}{a_1 x_1(0)^s e^{s \int_0^t \hat{x}_1(t')\, dt'} + a_2 x_2(0)^s e^{s \int_0^t \hat{x}_2(t')\, dt'}} \\ &= \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} \frac{e^{s \int_0^t \hat{x}_1(t')\, dt'}}{e^{s \int_0^t \hat{x}_2(t')\, dt'}} + 1} &&= \frac{1}{\frac{a_1 x_1(0)^s}{a_2 x_2(0)^s} e^{-s \int_0^t \hat{x}_2(t') - \hat{x}_1(t')\, dt'} + 1} \ . \end{aligned}$$

References

Arrow, K. (1999), Discounting, Morality and Gaming, in Portney and Weyant (1999):13-21.

Arrow, K., H. Chenery, B. Minhas and R. Solow (1961), Capital–Labor Substitution and Economic Efficiency, The Review of Economics and Statistics 43(3):225-248.

Arrow, K., W. Cline, K.-G.Maler, M. Munasinghe, R. Squitieri and J. Stiglitz (1995), Intertemporal Equity, Discounting, and Economic Efficiency, in: J.P. Bruce, H. Lee and E. Haites [ed.]: 128-44, Cambridge University Press, Cambridge.

Azfar, O. (1999), Rationalizing Hyperbolic Discounting, Journal of Economic

Behavior and Organization 38:245-252.

Barro, R.J. and X. Sala-i-Martin (1995), Economic Growth, McGraw-Hill, New York.

Chiang, A. (1992), Elements of Dynamic Optimization, McGraw-Hill, New York.

Chichilnisky, G. (1996), An Axiomatic Approach to Sustainable Development, Social Choice and Welfare 13:231-257.

Coto-Millán, Pablo (1999), Utility and Production, Physica, Heidelberg.

Fisher, A. and J. Krutilla (1975), Resource Conservation, Environmental Preservation, and the Rate of Discount, Quarterly Journal of Economics 89(3):358-370.

Frederick, S., G. Loewenstein and T. O'Donoghue (2002), Time Discounting and Time Preference: A Critical Review, Journal of Economic Literature XL:351-401.

Goldstein, H. (1980), Classical Mechanics, Addison-Wesley, Reading, 2^{nd} edition.

Gollier, C. (2002), Time Horizon and the Discount Rate, Journal of Economic Theory 75:229-253.

Hanley, N., J.F. Shogren and B. White (1997), Environmental Economics in Theory and Practice, MacMillan Press, London.

HM Treasury, Great Britain (2003), The Green Book – Appraisal and Evaluation in Central Government, HM Treasury, London.

Kamien, M. and N. Schwartz (2000), Dynamic Optimization, Elsevier, Amsterdam, 2^{nd} edition.

Koopmans, T. (1960), Stationary Ordinal Utility and Impatience, Econometrica 28:287-309.

Laibson, D. (1997), Golden Eggs and Hyperbolic Discounting, The Quaterly Journal of Economics 112(2):443-477.

Li, C. and K. Löfgren (2000), Renewable Resources and Economic Sustainability: A dynamic analysis with heterogeneous time preferences, Journal of Environmental Economics and Management 40:236-250.

Lind, R. [ed.](1982), Discounting for Time and Risk in Energy Policy, Resources for the Future, Washington.

Malinvaud, E. (1974), Lectures on Microeconomic Theory, North-Holland, Amsterdam, 3^{rd} printing.

Newell, R.G. and W.A. Pizer (2003), Discounting the distant future: how much do uncertain rates increase valuations?, Journal of Environmental Economics and Management 46:52-71.

Pearce, D., B. Groom, C. Hepburn and P. Koundouri (2003), Valuing the Future: Recent advances in social discounting, World Economics 4(2):121-141.

Phelps, E. and Pollak (1968), On Second-Best National Saving and Game–Equilibrium Growth, Review of Economic Studies 35:185-99.

Portney, P. and J. Weyant [ed.](1999), Discounting and Intergenerational Equity, Resources for the Future, Washington.

Sakurai, J.J. (1985) Modern quantum mechanics, Addison-Wesley, Reading.

Samuelson, P. (1937), A Note on Measurement of Utility, Review of Economic Studies 4:155-161.

Takayama, A. (1994), Mathematical Economics, Cambridge University Press, Cambridge, 2^{nd} edition, 8^{th} reprint.

Weitzman, M. (1994), On the "Environmental" Discount Rate, Journal of Environmental Economics and Management 26:200-209.

Weitzman, M. (1998), Why the Far-Distant Future Should Be Discounted at Its Lowest Possible Rate, Journal of Environmental Economics and Management 36:201-208.