Multiple Nutrient Stocks and Sustainable Agriculture

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Abstract

This paper develops a basic dynamic economic model that can be used for theoretical and numerical analysis of optimal soil management practices. A dynamic biophysical/economic optimal control model is developed in a multi-disciplinary framework, treating soil as a multi-pool portfolio of a particular limiting mobile nutrient (e.g. nitrogen). This specification allows for fertilizer to directly enter the active pool, while tillage initially affects the decadal pool, reflecting the realities of agricultural production. We examine the properties of the steady-state and the time paths of the optimal solutions, as well as exploring relevant comparative statics for the stationary point. In addition, alternative sustainability criteria of farm-level agricultural practices are presented, and the optimal solution of the problem is evaluated to determine if it meets any or all of the definitions of sustainability.

Keywords: Sustainability, nutrient management, renewable resources, optimal control

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1. Introduction

Concerns over the degradation of agricultural land and externalities associated with agricultural production have recently lead to studies of so-called “sustainable agriculture”, especially in the fields of agronomy and soil science. Usually utilizing replicated field trials and researcher-selected management practices, these studies have examined biological and chemical responses to alternative management regimes, including low inorganic input farming, cover cropping, and disparate tillage practices. While certainly instructive from a biogeochemical standpoint, there is a need to incorporate these ideas into an economic model of agent behavior in order to analyze, and in some cases predict, the behavior of individual growers when faced with a set of economic incentives.

Many economic models have been developed to analyze optimal soil management strategies, beginning with those of McConnell (1983) and Barrett (1991). These models recognized that the problem of optimal soil management was dynamic in nature, and used optimal control theory in order to analyze optimal behavior. Extensions to the basic dynamic model appearing in the literature include a more realistic representation of specific biophysical processes (Seppelt 1999), inclusion of multi-period investment variables to account for capital stocks (Grepperud 1997), and explicit modeling of the choices through which the farmer optimizes his profits through indirect manipulation of the stock of the soil natural resource, including input utilization (LaFrance 1992; Barrett 1991; Krautkraemer 1994; Brekke, et. al 1999; Hoag 1998) and choice of cropping system (Goetz 1997).

To date, however, these models have included relatively simple representations of soil quality via a single state index representing productivity, essentially ignoring the nutrient cycles now generally accepted and utilized in agronomic simulation modeling (see Baisden and Amundson 2003 for details). As the sustainable agriculture movement seeks to find management practices that can be used to affect these cycles, a multiple pool structure of nutrient availability is an imperative. In fact, a more realistic biogeochemical representation of the nutrient cycling process has important implications for the qualitative characterization of the optimal fertilization path and total nitrogen stocks. Few if any analytical dynamic economic optimization models have been constructed that have this capability, and even fewer have examined optimal behavior under different sustainability paradigms (one notable exception is Richter and Seppelt 1996).

As such, this paper provides a basic model that can be used for theoretical and numerical analysis of the sustainability rules of farm-level agricultural practices under alternative definitions of the concept, including the Rawlsian notion of a constant aggregate consumption (or utility) level known as the "maximin" criterion and the "wealth-constant" criterion put forth by Hicks in 1939 (Hartwick 1977; Solow 1986; Farzin 2002; Farzin 2004). A dynamic biophysical/economic optimal control model is developed in a multi-disciplinary framework, treating soil as a multi-pool portfolio of a particular limiting mobile nutrient (e.g. nitrogen). This specification allows for fertilizer to directly enter the active pool, while tillage initially affects the decadal pool, reflecting the realities of agricultural production. We examine the properties of the steady-state, as well as exploring relevant comparative statics for the stationary point. Several sustainability criteria are proposed, and the optimal paths are evaluated in the context of each definition. For simplicity, we restrict attention to interior solutions, essentially ruling out cases where initial nutrient stock levels are high relative to their respective steady state values. Upper bounds on fertilization, through regulation or leaching, is also assumed away for simplicity; as such, any generalizations to the results should be made with caution.

This paper is organized as follows. Section 2 describes and analyzes the general utilitarian model, including a characterization in both mathematical and graphical terms of the optimal solution to the problem. Section 3 develops several alternative sustainability criteria in the context of the general model, and determines if the optimal solution satisfies the particular definition under consideration. Section 4 discusses the comparative statics of the steady state of the system with respect to the relevant economic parameters of the problem, such as output price, input cost, the discount rate, and the technological parameters on the yield function. Section 5 concludes.

2. The General Model: An Economic Biogeochemical Representation

2.1 *A Model of Nutrient Cycling*

Our general model is an adaptation of the long-term terrestrial-ecosystem biogeochemistry model presented in Baisden and Amundson (2003), referred to here as the BA model. The BA model is a relatively simple analytical representation of the structure of popular (and considerably more complex) dynamic simulation models, such as Century and EPIC, that account for the interaction between plants and soil carbon and/or nutrient flows between storage compartments. Assuming three pools of a yield-limiting nutrient (nitrogen) and one pool of both above- and below-ground plant biomass[1], the model can be written as the following system of differential equations:

$$\frac{d}{dt}\mathbf{N} = \mathbf{JN} + \mathbf{B}, \tag{1}$$

where **N** is a 4x1 vector of nutrient pools, **J** is the 4x4 matrix of transfer coefficients that describes movement between the pools, and **B** denotes the vector of exogenous nutrient inputs or outputs of the system. The first row of **N** represents storage of nitrogen in plant biomass, while the last three rows correspond to pools of nitrogen that turn over on an approximate annual, decadal, and millennial scale. We detail our adaptation of this general model below; for more information on the BA model and how it is estimated, see Baisden and Amundson (2003), and Baisden, et al. (2002a, 2002b).

In order to embody the appropriate biogeochemical structure in an analytical economic optimization framework suitable for our analysis, we explicitly represent the evolution of two storage compartments, or stocks, of nitrogen in a manner almost identical to rows two and three of (1). Before turning to the full specification, however, it is instructive to trace through the conceptual framework of the model, beginning with the exogenous entry of a unit (kg/ha/yr) of nitrogen into the system via atmospheric deposition, denoted by γ. This unit first enters the "active" nitrogen pool N_1, so called because it turns over (i.e., gets released into other portions of the model) approximately every year. Denote the parameter that governs this turnover by $k_1 \approx 1$, so that $k_1\gamma$ of the initial deposition leaves this pool within a year. Of this $k_1\gamma$, a fraction ρ_2 ($.08 < \rho_2 < .36$ in the BA model) enters the decadal pool of nitrogen, denoted N_2, another fraction ρ_3 ($.001 < \rho_3 < .007$) enters the millennial pool, and the remainder ($1- \rho_2 - \rho_3$) enters the available supply of nitrogen to any potential vegetation. More generally, the entire pre-existing stock of N_1 turns over at the same rate k_1, and follows the same pathways, with the majority entering the available supply every year. The second stock N_2 turns over as well, with k_2 ($.02 < k_2 < .08$) of the existing decadal stock released to the available supply every year through natural decomposition processes. The release from this pool can be accelerated through tillage practices, denoted $1 \le T < (k_1 / k_2)$, so that the fraction of total release from the decadal pool is $k_2 T$ for a given tillage regime. Because the millennial pool essentially does not turn over under most scenarios, the dynamics of the millennial pool are not explicitly modeled, and as such $\rho_3 k_1 N_1$ is exported directly from the active pool each year.

As alluded to above, nitrogen in plant biomass is not directly modeled as a stock, but rather the nitrogen available to any vegetation is calculated as a flow measure. This supply, denoted N_a, consists of a linear combination of the explicitly represented state variables N_1 and N_2, taking into account exports via leaching, erosion, denitrification, etc… and additional supply sources such as fertilizer and the input from the exogenous millennial pool. Specifically, define

$$N_a(t) = (1-\eta)\left[(1-\rho_2-\rho_3)k_1 N_1(t) + k_2 T N_2(t) + \gamma_3 + \gamma_{atm} F(t)\right] \quad (2)$$

where η represents the export rate out of the system via the above processes, γ_3 denotes inputs from the excluded pool, $F(t)$ characterizes fertilizer inputs as a multiplicative factor of the natural rate of atmospheric deposition γ_{atm}, and all other variables are as defined above. Note that fertilization is subject to the same export rate as the other sources of nitrogen, and that fertilizer is a perfect substitute for indigenous nitrogen.[2]

Finally, in order to complete the nitrogen cycle, the return of any plant biomass and/or unused nitrogen from the available supply to the active pool is represented. Following the BA model, denote a harvest index H, $0 \leq H \leq 1$, to represent the proportion of available supply, in nutrient units, extracted through harvest of crop material, implying $(1-\eta)(1-H)N_a$ is returned to the active pool each year. If, for example, the land is left fallow, H=0. If a crop or sequence of crops is cultivated, then H is specific to the biological realities of the crop and its nutrient use efficiency. Baisden and Amundson (2003) assume that $.2 \leq H \leq .5$ in explaining the effect of agriculture on the natural system. With these assumptions, the dynamics of the nitrogen cycle can generally be represented as

$$\dot{N}_1 = \left[(1-\eta)(1-H)(1-\rho_2-\rho_3)-1\right]k_1 N_1 + (1-\eta)(1-H)\left[k_2 T N_2 + (\gamma_3 + \gamma_{atm} F)\right] + \gamma_{atm} \quad (3)$$

$$\dot{N}_2 = \rho_2 k_1 N_1 - k_2 T N_2 \quad (4)$$

which, when taken with (2), essentially corresponds to the first three rows of the BA model presented in (1). We thus have a system of two simple linear differential equations that generally define the movement of a limiting nutrient through the biogeochemical process. A schematic representation of the nutrient cycle is presented in Figure 1.

2.2 *A Model of Economic Behavior*

One of the primary contributions of this paper is to integrate this general analytical description of nutrient cycling, which is at the core of many agricultural simulation models, with the traditional behavioral assumptions of economics. In particular, we assume a sole agent who owns a normalized unit of land which admits a nitrogen cycle described by equations (2) - (4). This (risk-neutral) agent seeks to maximize the stream of his profits over an infinite time horizon by cultivating a crop or sequence of crops with given harvest index *H* and associated tillage system *T*, with fertilizer application as a choice variable. In more formal terms, the agent solves the problem (suppressing *t* as an argument for available nitrogen and fertilizer)

$$\max_{F \geq 0} \int_0^\infty [pY(N_a; H, T) - cF] e^{-rt} dt \tag{5}$$

subject to (2) - (4), and the initial conditions and non-negativity constraints given by $N_1(0) = N_{10} > 0, N_2(0) = N_{20} > 0, N_1(t) \geq 0,$ and $N_2(t) \geq 0.$ We define *p* as the relative price of one unit of yield, defined by the production function $Y(N_a)$, while *c* is the relative price of fertilizer in the appropriate units. We assume that *H* and *T* exogenously and jointly determine the structure of $Y(N_a; H,T)$, and $Y'(N_a)>0$, $Y''(N_a)<0$, and $Y(0)=0$.

Several points regarding these assumptions and the structure of the model are worth mentioning. First, this specification can be described as a hybrid between a traditional renewable resource problem and a traditional investment problem, both in a dynamic setting. From the point of view of a renewable resource problem, harvest *H* depletes a natural resource stock $N = N_1 + N_2$ which evolves naturally over time and in the absence of anthropogenic activity, tends towards a steady-state. In the model described in this paper, however, the conventional "effort" (as measured by the harvest index *H*) is not a choice variable, nor does one directly consume and accrue benefits from exploitation of the resource. Rather, *H* is determined through crop choice, and one can view the choice of the tandem *H* and *T* (and thus the associated yield function) as occurring prior to fertilization choice, and fixed throughout the time horizon. This is obviously a gross simplification, especially with regard to the fixed nature of these choices over time, and can be relaxed (see Bond 2004), but we restrict attention here to the specification in (5) for simplicity. Viewing the problem as an investment problem, the agent purchases stock-augmenting flows (i.e., investment) in each time period, which in turn add to stock levels subsequently used in the production process. In this sense, we capture the indirect nature of the relationship between the state variables and the welfare function. Note that unlike some firm investment models, there are irreversibilities in investment, in that one can only augment, rather than deplete, the nutrient stocks through direct manipulation of the control. In other words, we are constraining $F(t)$ to be positive over the planning horizon, and there is can be no "disinvestment" in nutrient stocks available due to the prohibitive costs of doing so. A value of $F(t)=0$ with a harvest index greater than zero implies cultivation without any fertilization.

Second, we have assumed constant relative prices over the time horizon. While this may be troublesome if the model were extended over a larger geographic area, thus introducing endogeneity into price determination, we argue that the single price-taking agent specification minimizes any potential errors given the necessary analytical simplifications. Of course, different expectations over future prices can (and most likely will) alter the optimal investment strategies. Lastly, we have specified an infinite terminal time, largely as a result of the fact that the problem makes economic sense only if a steady-state solution to the problem exists.

We now turn to the necessary and sufficient conditions that must be satisfied for a solution to (2) - (5) and the initial and terminal conditions. Define the current-value Hamiltonian as

$$\begin{aligned} H^1(F,N_1,N_2,\lambda_1,\lambda_2;H,T) = {} & pY\left(N_a(N_1,N_2,F);H,T\right) - cF \\ & + \lambda_1\left[a_{11}N_1 + a_{12}N_2 + (1-\eta)(1-H)(\gamma_3+\gamma_{atm}F)+\gamma_{atm}\right] \\ & + \lambda_2\left[a_{21}N_1 + a_{22}N_2\right] \end{aligned} \tag{6}$$

where λ_1 and λ_2 are the current-value costate variables of N_1 and N_2, respectively, reflecting the shadow values of the two stocks, and the a_{ij}'s (i,j=1,2) are the coefficients on N_1 and N_2 in (3) and (4).[3] Defining the Lagrangian as

$$L(F,N_1,N_2,\lambda_1,\lambda_2,\mu;H,T) = H^1(F,N_1,N_2,\lambda_1,\lambda_2;H,T) + \mu F \tag{7}$$

where μ is the shadow value associated with the non-negativity constraint on F, Theorem 36.3 of Caputo (2004) gives the necessary conditions for an optimal solution:

$$L_F = (1-\eta)\left[pY'(N_a)\gamma_{atm} + \lambda_1(1-H)\gamma_{atm}\right] - c + \mu = 0;\ F \geq 0,\ \mu \geq 0,\ \ \mu F = 0 \tag{8}$$

$$\dot{\lambda}_1 = \left(r - a_{11}\right)\lambda_1 - pY'(N_a)\frac{(a_{11}+k_1)}{(1-H)} - a_{21}\lambda_2 \tag{9}$$

$$\dot{\lambda}_2 = (r - a_{22})\lambda_2 - pY'(N_a)\frac{a_{12}}{(1-H)} - a_{12}\lambda_1 \tag{10}$$

$$\dot{N}_1 = \left[(1-\eta)(1-H)(1-\rho_2-\rho_3)-1\right]k_1N_1 + (1-\eta)(1-H)\left[k_2TN_2 + (\gamma_3+\gamma_{atm}F)\right] + \gamma_{atm} \tag{11}$$

$$\dot{N}_2 = \rho_2 k_1 N_1 - k_2 T N_2 \tag{12}$$

$$\lim_{t\to\infty} N_i \geq 0,\ i = 1, 2. \tag{13}$$

We will restrict attention in this paper to the case of an interior solution; i.e., μ=0, although it should be noted that there are some interesting cases when the non-negativity constraint binds at some point over the planning horizon (see Appendix A for conditions that guarantee an interior solution, and Bond (2004) for further details). As such, the optimal solution will lead to a steady state. Sufficiency conditions are satisfied as well, as Theorem 36.4 of Caputo (2004) can be used because for any admissible control F and associated state path N_j, L is concave in (F, N_1, N_2), and $\lim_{t\to\infty} \sum_{j=1}^{2} \lambda_j \left[N_j^* - N_j \right] \leq 0,$ where N_j^* is the optimal state path associated with any optimal control F^* that satisfies (8)-(13). Thus, any solution to the necessary conditions is an optimal control, although this theorem does not guarantee uniqueness.

Interpretation of the necessary conditions is relatively straightforward. Equation (8) states that the marginal value generated by application of a unit of fertilizer, in terms of both *immediate* benefits through crop growth and harvesting and *indirect* benefits through the unused portion returned to the nitrogen system, must equal the marginal cost of fertilizer at each moment in time. Equations (11) and (12) just restate the structure of the system, and (13) is the transversality condition that ensures non-negativity of the nitrogen stocks. From (9) and (10), each costate variable (or the shadow price associated with each pool of nitrogen), must grow at the rate of discount less the marginal value lost (gained) in each pool through export (import) of both harvesting and natural processes. Because fertilizer and indigenous nutrients are perfect substitutes, we are assuming an interior solution, and there is no upper bound on fertilizer use, the reader might already suspect that the costate variables are constant over time, a result we will confirm for a special case (yet without loss of generality) in the next section.

2.3 *Solution to a Linear-Quadratic Approximation*

We now turn to the characterization of the optimal solution. In order to more fully depict the solution to the problem under consideration, the yield function is approximated by the quadratic equation $Y(N_a; H, T) = bN_a - dN_a^{\ 2}$, with the understanding that H and T, in part, implicitly determine the coefficients b and d. This approximation is not necessary in order to apply the methodology applied here; however, it does ensure that the results are global, rather than local around a neighborhood of the steady state. The reader is reminded that we are restricting attention to interior solutions.

First, it is instructive to examine the question of existence and stability of the steady state. From (8), an expression for F in terms of the state and costate variables can be derived (and is given in Appendix A). Denoting this function $F(N_1, N_2, \lambda_1)$, and substituting into the four differential equations above, the modified Hamiltonian dynamic system (MHDS) can be written as:

$$\dot{N}_1 = -k_1 N_1 + \frac{(1-H)^2}{2dp}\lambda_1 + \frac{\gamma_{atm}(1-\eta)\left[(1-H)bp + 2dp\gamma_{atm}\right] + (1-H)c}{(1-\eta)2dp\gamma_{atm}} \tag{14}$$

$$\dot{N}_2 = k_1\rho_2 N_1 - k_2 T N_2 \tag{15}$$

$$\dot{\lambda}_1 = (k_1 + r)\lambda_1 - k_1\rho_2\lambda_2 - \frac{ck_1(1-\rho_2-\rho_3)}{\gamma_{atm}} \tag{16}$$

$$\dot{\lambda}_2 = (r + k_2 T)\lambda_2 - \frac{ck_2 T}{\gamma_{atm}}. \tag{17}$$

The dynamics of this system of linear differential equations are governed by the Jacobian of the system, denoted here by the 4x4 matrix **J**, with typical element $\frac{\partial \dot{X}}{\partial z}; X, z \in (N_1, N_2, \lambda_1, \lambda_2)$. Because the determinant of **J**≠0 and the system is linear, the steady-state exits, and, in fact, is unique (see Appendix B). Furthermore, it can be shown that the determinant is positive, and satisfies the conditions of Theorem 3 in Dockner (1985) (see Appendix C). The stationary point is thus a saddle point, and exhibits a two-dimensional stable plane on which all paths asymptotically approach the steady state (Tahvonen 1991). At least one of these paths is the solution to the problem.

We can identify this path by recognizing that along the stable manifold, the solution to this system, in general, can be expressed as

$$\boldsymbol{\varphi}(t; \mathbf{x}_\infty, \mathbf{x_0}, \mathbf{A}) = \mathbf{x}_\infty + c_1\mathbf{v^1}e^{r_1 t} + c_2\mathbf{v^2}e^{r_2 t} \tag{18}$$

where $\boldsymbol{\varphi}(\cdot) = \left(N_1(t) \quad N_2(t) \quad \lambda_1(t) \quad \lambda_2(t)\right)'$, $\mathbf{v}^i$, i=1,2 are the 4x1 eigenvectors of **J**, r_i are the negative eigenvalues of **J**, c_i are constants that are determined by the initial conditions $\mathbf{x_0} = \{N_{10} \quad N_{20}\}$, and $\mathbf{x}_\infty$ are the steady state values of the system. Appendix D details these calculations. The specific solution to the problem is thus

$$N_1(t) = N_{1\infty} + (N_{10} - N_{1\infty})e^{-k_1 t} \tag{19}$$

$$N_2(t) = N_{2\infty} + \frac{k_1\rho_2}{-k_1 + k_2T}(N_{10} - N_{1\infty})e^{-k_1 t} + \left(N_{20} - N_{2\infty} + \frac{k_1\rho_2}{-k_1 + k_2T}(N_{1\infty} - N_{10})\right)e^{(-k_2T)t} \tag{20}$$

$$\lambda_1 = \lambda_{1\infty} \tag{21}$$

$$\lambda_2 = \lambda_{2\infty}. \tag{22}$$

Substituting these values into the equation for F,[4] the optimal control is

$$\begin{aligned} F(t, N_1, N_2, \lambda_{1\infty}) = &-\frac{k_1(1-\rho_2-\rho_3)}{\gamma_{atm}}N_1(t) - \frac{k_2T}{\gamma_{atm}}N_2(t) + \frac{(1-H)}{(1-\eta)2pd\gamma_{atm}}\lambda_{1\infty} \\ &+ \frac{(1-\eta)p\gamma_{atm}(b-(1-\eta)2d\gamma_3) - c}{(1-\eta)^2 2pd\gamma_{atm}^2}. \end{aligned} \tag{23}$$

As can be seen in (23), optimal fertilizer application is inversely related to both nitrogen stocks, as should be expected. Similarly, an increase in the marginal value of the active nitrogen pool through a change in one of the relevant problem parameters (e.g. c, r, or T) increases the marginal benefit of fertilizer, resulting in a greater application rate. We now discuss the optimal paths of the state and control variables, which are largely determined by the initial stock values.

PROPOSITION 1: *The optimal time path of the active nutrient stock N_1 is monotonic, and its direction is solely determined by the initial stock level.*

Proof: Differentiate (19) with respect to time t, and note that $\text{sign}\,(\dot{N}_1) = -\text{sign}(N_{10} - N_{1\infty})$. ■

PROPOSITION 2: *The optimal time path of the decadal nutrient stock N_2 is not necessarily monotonic, even if the initial stock levels of both N_1 and N_2 are greater or less than their respective steady state values, but can change direction only once.*

Proof: Begin by differentiating (20) with respect to time to obtain

$$\dot{N}_2(t) = -\frac{k_1^2 \rho_2}{k_2 T - k_1}(N_{10} - N_{1\infty})e^{-k_1 t} - k_2 T\left[(N_{20} - N_{2\infty}) - \frac{k_1 \rho_2}{k_2 T - k_1}(N_{10} - N_{1\infty})\right]e^{-(k_2 T)t}. \quad (24)$$

Note that because we have assumed $T < k_1/k_2$, the term $k_2 T - k_1 < 0$. As such, the first and third additive terms in (24), once expanded, have opposite signs, with the first term greater in absolute value than the third at time t=0. However, given the assumption on T, the first term will be dominated by the third after some $t = \tau > 0$. The sign of the time path is thus determined by the second term, $-k_2 T(N_{20} - N_{2\infty})e^{-(k_2 T)t}$, which has the same sign as the third term if the signs on $(N_{i0} - N_{i\infty})$ are identical, but the opposite sign otherwise. If this term is relatively large in absolute value, i.e., if the initial value of the N_2 is relatively far from its steady state value, then the time path of N_2 is monotonic; however, if it is relatively small in magnitude, then the sign of (24) will change exactly once, at time t=τ, and the time path of this stock will switch direction and be either U-shaped or inverted-U shaped. ■

The fact that a nutrient pools can exhibit a non-monotonic optimal path is a feature of the multiple state variables in the model, and one that cannot occur with models incorporating only one state variable. A similar result regarding global carbon cycles in the context of carbon accumulation in the atmosphere can be found in Farzin and Tahvonen (1996).

Finally, we turn to the characterization of the optimal path for fertilization.

PROPOSITION 3: *The optimal time path of the fertilizer control F is monotonic if the initial stock levels of both N_1 and N_2 are greater than or less than their respective steady state values. If, however, one of the initial levels is greater than its respective steady state value and the other is less, the optimal fertilization time path will either be monotonic or switch direction exactly once.*

Proof: The proof for Proposition 3 is similar to that of Proposition 2. Substitute (19) and (20) into (23), differentiate with respect to t, and collect like terms to obtain

$$\dot{F}(t) = \left[\frac{k_1^2(1-\rho_2-\rho_3)}{\gamma_{atm}}e^{-k_1 t} - \frac{k_1 k_2 T \rho_2}{\gamma_{atm}}e^{-(k_2 T)t}\right](N_{10} - N_{1\infty}) + \frac{(k_2 T)^2}{\gamma_{atm}}e^{-(k_2 T)t}(N_{20} - N_{2\infty}). \quad (25)$$

Note that the coefficients on each $(N_{i0} - N_{i\infty})$ term are positive under the assumptions of this paper. As such, the sign of $\dot{F}(t)$ is entirely determined by magnitude and sign of the difference between initial and steady state stock values, and in the case of different signs, the time at which it is evaluated. Obviously, the sign of (25) does not change if both stocks begin either greater than or less than the steady state values, thus implying monotonicity of the time path for fertilization. On the other hand, if one of the initial stock values is greater than and one is less than their steady state values, $\dot{F}$ can change direction at some point $t = \tau > 0$, provided the difference between the initial value and steady state for N_1 is "large enough". To see this, note that the first term in (25) will decline more rapidly than the second and third term as time evolves, given our assumptions on T. If first term is greater in magnitude than the sum of the second and third term at time t=0, then there exists some $t = \tau > 0$ such that the values are equal. The derivative of the optimal fertilization path will thus change sign at τ, and thus the time path will switch from increasing to decreasing, or vice-versa. ■

The dependence of each nutrient stock level on the other thus allows for a U-shaped or inverted U-shaped optimal fertilization schedule. For example, if N_1 is low and N_2 is high initially, it may be optimal to directly substitute for the active pool via decreasing, but positive, fertilization levels in the beginning of the planning horizon, building up the stocks of the decadal pool as well. As a result of leaching and crop export, however, these gains are eventually diminished, and increasing fertilization levels are possible. Again, a model with one state variable and an infinite time horizon cannot admit an optimal fertilization schedule that is non-monotonic. Thus, a more realistic biogeochemical representation of the nutrient cycling process has important implications for the qualitative characterization of the optimal fertilization path and total nitrogen stocks.

A graphical representation of these concepts in given in Figure 2, which depicts phase portraits along the optimal stable manifold.[5] Figure 2a shows the relationship between nutrient stock levels, and confirms qualitatively the propositions proved above. The diagonal dotted line depicts the non-negativity constraint, in that any starting values to the left of the line are admissible under the assumption of an interior solution. Using this information, it is clear that the non-negativity constraint is binding primarily in situations when initial nutrient levels in *both* pools are relatively high, or the relative distance from the steady state for one pool is considerably higher than the other. We would expect initial values such as these to be representative of undisturbed land not previously cultivated. Note that along the stable manifold, the admissible paths to the steady state can be described as a stable node, with monotonic paths for N_1 regardless of the level for N_2. Furthermore, for these parameter values, it can be seen that unless N_{20} is very close to the steady state level, it tends to be monotonic as well.

Figures 2b and 2c represent the same paths as those in 1a, but with fertilization on the vertical axis and one stock on the horizontal axis. As these graphs are relatively difficult to interpret, it is important for the reader to recognize that the paths depicted here are conditional on the starting values of the stock not represented in the graph. To see the relationship between the three graphs, one particular path labeled "a" has been identified. Figures 2b and 2c graphically display the fact that the optimal fertilization schedules are much more likely to be non-monotonic in nature than N_2, a fact which will be quite important when we examine the alternative sustainability criteria in the next section.

3. Sustainability Criteria and the Economic Biogeochemical Model

We now turn to a discussion of this model in terms of alternative notions of sustainable agriculture, using the macroeconomic growth literature as a guide. Pezzy (1997) summarizes several alternative sustainability concepts in terms of constraints that could be placed on a present value optimization problem like the one described in equations (2) - (5), and we use these, as well as the concepts of constant aggregate welfare level known as the Rawlsian "maximin" criterion and the "wealth-constant" criterion put forth by Hicks to examine the sustainability properties of the optimal solutions derived in Section 2 (Solow 1974; Hartwick 1977; Solow 1986; Farzin 2004). For each criterion, we wish to know if the utilitarian optimal solution satisfies the particular definition of sustainability, and if not, precisely where it fails to do so.

For each criterion under consideration, we first formally define the notion of sustainability, and then subsequently analyze the optimal solution to answer the question of satisfaction. We examine each in turn.

DEFINITION: An optimal path is "strongly sustainable", or equivalently "ecologically sustainable", if the sum of total nutrient levels across the stock pools does not decline throughout the planning horizon.

This is the criterion that is the most restrictive in a purely physical sense, in that it implies that $\dot{N}(t) = \dot{N}_1(t) + \dot{N}_2(t) \geq 0 \; \forall t \in [0, \infty).$ In other words, the initial stock levels are (at least) maintained indefinitely, *with no decline in total nutrient stocks allowed at any point over the time horizon.* From an intergenerational equity point of view, this implies that every subsequent generation has at least the same total physical level of the nutrient stocks available for production as did the immediately preceding generation. We are thus not necessarily concerned with an economic welfare measure under this criterion, but rather an ecological measure. Note that this does not require, however, that each individual stock must satisfy $\dot{N}_i \geq 0,\ i$=1,2, as some radical ecologists might favor.

To evaluate the optimal solution under strong sustainability, differentiate (19) and use (24) to obtain

$$\dot{N}_1 + \dot{N}_2 = \left[-k_1 \left(\frac{k_1\rho_2}{k_2T - k_1} + 1 \right) e^{-k_1 t} + \frac{k_2Tk_1\rho_2}{k_2T - k_1} e^{-(k_2T)t} \right] (N_{10} - N_{1\infty}) - k_2Te^{-(k_2T)t}(N_{20} - N_{2\infty}). \quad (26)$$

As both of the coefficient terms on $(N_{i0} - N_{i\infty})$ are non-positive and declining in t, it should be obvious that a sufficient condition for the optimal solution to satisfy this criterion is that the initial value for each N_i is less than the steady state value. Thus, every path to the southwest of the steady state in Figure 2a is strongly sustainable under the definition. However, it is *not* a necessary condition, as the different decay rates given by the eigenvalues $-k_1$ and $-k_2T$ allow for the possibility that the starting values can be on opposite sides of the steady state and yet still satisfy the definition. In other words, although one of the nutrient stocks might be declining over a particular subset of the time horizon, the total nutrient stocks may be increasing. In any of these cases, however, the initial conditions determine the status of the optimal solution under the criterion.

It is also noteworthy to recognize that a similar criterion would be the condition that $N(t) = N_1(t) + N_2(t) \geq N_{10} + N_{20} = N_0\ \forall t.$ In this case, the aggregate pool is allowed to decline over some period of time, but only if the stock was first increased through investment, and not to the extent that it ever dips below the initial levels. A simple way to evaluate if the optimal solution satisfies this modified ecological criterion is to define the line $N_2(t) = N_{10} + N_{20} - N_1(t)$ and graph it on Figure 2a. Any path that lies continuously to the right of this line satisfies the criterion; in the case depicted, most of the paths with relatively small initial stocks of N_2 fulfill the condition, while those with relatively larger N_2 stocks at time 0 do not. However, in terms of the implications for welfare across generations, we have essentially arbitrarily chosen the first generation as a benchmark, with no real ethical justification for this choice. Furthermore, there would be no need to distinguish between the modification and the original definition in the case of one state variable, as the saddle point property in two dimensions would ensure monotonicity, and thus equivalence of the criterion.

DEFINITION: An optimal path is "yield sustainable" if the time path of yield does not decline at any time throughout the planning horizon.

Yield sustainability, as defined here, is essentially a bridge between a physical concept and an economic concept, because the actual object to be sustained is still physical in nature, but the sole source of revenue in the model. As output price does not change over time, this idea could be called "revenue sustainability" as well. Formally, we define this criterion as $\dot{Y}(t) = Y'(N_a)\dot{N}_a \geq 0.$ Recall that we assumed that over the relevant range of nutrient availability, the marginal product of nutrients is positive;[6] thus, this condition can be written as $\dot{N}_a \geq 0.$ By using the expression obtained for F from the first order condition (8) and substituting into (2), available nitrogen can be expressed as

$$N_a = \frac{-c + (1-\eta)\gamma_{atm}\left(bp + (1-H)\lambda_1\right)}{(1-\eta)2dp\gamma_{atm}}. \qquad (27)$$

But recall from (21) that the shadow values associated with each pool are constant, and thus $\dot{N}_a = 0 \ \forall t.$ Under this model structure, then, *any* optimal path is yield sustainable.

Several caveats need to be recognized at this point. First, we have not restricted fertilizer application to an upper bound, thus allowing for constant nitrogen availability over the planning horizon (and constant shadow values). This is quite realistic under slightly degraded conditions, for example, but unlikely to be possible if soil is severely degraded in terms of nutrient content. However, it is the driving force behind the conclusion that any optimal path is yield sustainable, as the optimizing agent essentially seeks to maintain N_a through the time horizon. The lower bound can also affect this conclusion, as high initial values may produce large, economically unsustainable yields due to the cost of fertilizer. However, in this case, such situations are ruled out. Second, we have assumed that there are no ill effects from continuous fertilizer usage, such as water pollution, that subsequently adversely affects yields in future periods. For more discussion about these non-negativity and externality issues, see Bond (2004).

DEFINITION: An optimal path is "profit sustainable" if the time path of profit does not decline at any time throughout the planning horizon.

Unlike the previous two criteria, profit sustainability is concerned with the time path of an economic welfare measure rather than a physical stock or flow. The profit function in the model is $\pi = pY(N_a) - cF,$ so that the time derivative is $\dot{\pi} = pY'(N_a)\dot{N}_a - c\dot{F} = -c\dot{F}$ using the result obtained from the yield sustainability criterion. Clearly, then, we require further analysis of the optimal fertilization schedules. Figure 3 depicts these schedules as a function of time, assuming a variety of starting values for both the active and decadal nitrogen pools, and splits them into paths which violate profit sustainability and those that satisfy it. The initial values, reported in the legend relative to the steady state (except for the severely degraded soil), are a subset of those shown in Figure 2. Mathematically, we can describe the slope of the path as

$$\dot{F} = -\frac{k_1^{\,2}(1-\rho_2-\rho_3)}{\gamma_{atm}}\dot{N}_1 - \frac{(k_2T)^2}{\gamma_{atm}}\dot{N}_2, \tag{28}$$

which is just a general restatement of (25). Recognizing that both coefficients on the time derivatives of nutrient stocks are negative, it is clear that decreasing stock levels over time is sufficient to violate the profit sustainability rule, as $\dot{F} < 0$ implies $\dot{\pi} > 0.$ This is also the case when one of the pools just happens to begin at the steady state level and the other begins above its respective stationary level and monotonically declines. The converse is true for increasing stock levels over time, as severely degraded soil offers the opportunity to use fertilization to augment natural deposition and restore fertility, and with profit levels low initially, profit sustainability is achievable. However, and perhaps most importantly, it is likely that if the initial stocks of nutrients are of mixed sign with respect to their distance from the stationary point, then $\dot{F}$ will change sign and the criterion will be violated. Again, this result cannot be achieved with a one-state model.

DEFINITION: An optimal path satisfies the Rawlsian "maximin" sustainability criterion if the profit level at each point in time is equal to the maximum constant instantaneous profit level possible.

This criterion, which has been much discussed in the macroeconomics growth literature, differs from profit sustainability in that it requires a degree of intergenerational equity (i.e., maximum *constant* profits over time) not essential under the definition of profit sustainability. This criterion suggests a version of the zero net aggregate investment rule, which states that the current value of changes in productive asset stocks at each point in time over the planning horizon should equal zero (Hartwick 1977; Solow 1986; Farzin 2004). Farzin (2002) shows that for any positive discount rate r>0, sustainability in the sense of constant utility (here, profits defined as $pY(N_a;H,T)-cF$) requires

$$\lambda_1\dot{N}_1+\lambda_2\dot{N}_2=0\ \forall t. \tag{29}$$

Thus, *in aggregate*, the value of the change in nutrients to the farm's productivity must be zero at each time in the planning horizon.

As is obvious from the previous discussion, it is quite unlikely that the optimal solution would satisfy (29), as the shadow values are constant and the time derivative of individual nutrient stocks can take virtually any sign, and not necessarily of offsetting magnitude. In general, then, the optimal solution does not admit the constant maximum profit level typical of intergenerational equality. This is not to say that such a solution does not exist, just that it is not optimal under the utilitarian paradigm. Such a stationary path would necessitate a loss of welfare over some subset of the time horizon by the definition of optimality, but the extent of this loss is not examined in this paper. For further analysis, see Bond (2004). However, if we allow (29) to be satisfied with an inequality, such that $\lambda_1\dot{N}_1+\lambda_2\dot{N}_2\geq 0,$ the value of net aggregate investment will not decrease over time, and thus provides the opportunity for future generations to be at least as well off in terms of profit as previous generations. Again, the optimal solution satisfies this modified constraint in the case of severely degraded soils with low initial starting values.

It is worth noting at this point that the Rawlsian maximin criterion introduces the notion of *value* of the nutrient stocks through the shadow values λ_i, as opposed to the other sustainability criterion that focus primarily on the value of flows alone. It is this difference that primarily separates the economic notions of sustainability from more traditional, and perhaps more familiar, definitions. In a more general case, such as a fertilization constrained problem, it is likely that the values of the stocks will change over the planning horizon (i.e., the λ_i will not be constant), so the investment rules that would result from imposing maximin sustainability would likely be more complicated.

DEFINITION: An optimal path is "stock value sustainable" if the value of the resource base is kept in tact over the time horizon.

The internal competitive valuation of any resource stock is given by the shadow value $\lambda_i(t)$, so at any point in time, the competitive value of the resource base is given by

$$V(t) = \lambda_1 N_1 + \lambda_2 N_2.$$

Farzin (2004) suggests that the maintenance of Hicksian income requires the time derivative of $V(t)$ be greater than or equal than zero, or formally,

$$\dot{V}(t) = \lambda_1 \dot{N}_1 + \lambda_2 \dot{N}_2 + \dot{\lambda}_1 N_1 + \dot{\lambda}_2 N_2 \geq 0. \tag{30}$$

Note that this measure takes into account not only the value of the change in nutrient stock levels, as in (29), but also the change in the value of the stock, or the capital gains from holding the nutrients in the soil (Farzin 2004). However, as previously discussed, the shadow values on each nutrient stock are constant over time, so the two criteria are identical in this case.

We have thus shown that in the purely renewable resource model presented here, assuming perfect substitutability between fertilizer and indigenous nutrients and no constraints on quantity of nutrients the actor can add to the soil, that four of the five sustainability constraints take the form

$$c_{1i} \dot{N}_1 + c_{2i} \dot{N}_2 \geq 0, \tag{31}$$

where the c_{ji} are positive coefficients of the i^{th} criterion on the time derivative of the j^{th} stock. Clearly, the initial values of each nutrient pool are critical to the utilitarian solution satisfying a particular sustainability rule. While not necessary, a sufficient condition for the optimal solution to be sustainable for each of the four is for the soil to be extremely degraded compared to the steady-state level of the stock. In the case of initial stock levels greater than the steady state considered here, "soil mining", defined as extracting the resource stocks faster than they can be replaced, is an optimal strategy. As seen above, this may or may not satisfy any of the sustainability criteria.

Furthermore, the yield sustainability criterion highlights the difficulty in selecting an appropriate sustainability rule. Particularly, is it the *availability* of a nutrient that must be sustained, which in this case ensures constant yields, or the *total* nutrient stock level defined by the sum of the stocks? *Any* optimal solution here satisfies the former, but not necessarily the latter or any of the other criteria, including the value of net investment, value of the entire farm, or profits over time. When considering policies associated with "sustainable agriculture", then, researchers should be especially vigilant in defining what exactly is to be sustained (a physical resource, a flow of physical resources, or a measure of welfare) over what time period (Pezzy 1997).

Given these results, we now turn to the comparative statics of the steady state.

4. Comparative Statics of the Steady State

With the sustainability criteria of the form shown in (31), it is useful to examine the economic parameters that could be manipulated by policy instruments in order to change the steady state level of nutrient stock levels. If, for example, one were to find an instrument, or combination of instruments, that would result in an increase in the stationary levels of both N_1 and $N_{2,}$ a larger subset of initial values could be considered sustainable under the criteria. In the case of yield sustainability, we would like to know the instruments that can be used to increase the steady state level of λ_1, and thus N_a. Additionally, technology has been assumed constant throughout the course of the paper thus far, though the effects of changing technology on sustainability and the steady state levels of nutrient stocks and fertilizer is of considerable interest.

Table 1 reports the comparative statics of the economic and technological parameters of the model (for computation, see Bond (2004)). The reader is cautioned that although the parameters b and d are implicitly influenced by H and T, these comparative statics assume separability between the parameters.

An increase in relative output price will ultimately not affect the value of nutrients in the soil, but will encourage a greater level of fertilization due to optimal nitrogen availability increasing (see (27)). The increased fertilization rate thus increases the steady state levels of both nitrogen stocks, and hence allows a greater subset of feasible initial values to be considered sustainable under the criteria presented here. However, for each initial set of values, the steady state, an ecologically sustainable point, is 'farther away", both in terms of stock levels and time. Recognize, however, that this result is specific to the assumptions we have made here, as optimal cropping intensity might be expected to increase in a more general model, resulting in downward pressure on nutrient stocks. Furthermore, we have ignored any externalities associated with fertilizer usage, and subsequent generalizations of the model should take these into account, thus influencing the comparative static result.

An increase in the relative cost of fertilizer has the anticipated effects of lowering optimal nutrient stocks and fertilizer usage, in accordance with the law of demand. We also note that such a change increases the marginal value of nutrients in the soil, potentially affecting, to a greater extend than the physical criteria, characterization of the economic sustainability rules. An increase in the discount rate, representing a preference towards current production at the possible expense of future generations, has the same effect on nutrient stocks but the opposite effect on shadow values, diminishing the value of holding nutrient stocks, and leading to incentives that are consistent with soil mining.

We now examine changes in the technological parameters of the production function. An increase in *b*, which is the coefficient on the linear portion of the production function, will increase the level of nitrogen that maximizes yields and increase that maximum yield level. After the change, yields for each nitrogen availability level are greater than the original. As such, the optimal response is to intensify fertilization to take advantage of this new production technology, thus leading to greater steady state nutrient stock levels. On the other hand, and increase in *d*, the quadratic term parameter, reduces the level of nitrogen necessary for maximal yield, and reduces the maximum yield level as well. Not surprisingly, then, we obtain results opposite to those of *b*. In reality, it is likely that a new technology, e.g. a new hybrid or genetic variety, would affect both parameters, as well as the harvest index, thus making the sign of a technology change indeterminate.

Finally, we examine the comparative statics of the harvest index H and the tillage index T. The harvest index is directly related to the export of nitrogen out of the system, so any increase will necessitate steady state stock levels less than before the change, with the expected decrease in fertilization rates. Tillage, on the other hand, accelerates release from the decadal pool, thus increasing the stock levels of N_1, but increases turnover of N_2. Note that this is the only manner in which a manager can actually change the natural biogeochemical dynamics of the soil. The sign on the comparative static on the decadal pool thus cannot be determined without specific values for the parameters. However, note that an increased tillage index increases the value of nutrients in the soil in both pools, yet increases optimal fertilizer rates as well. In subsequent extensions of this paper, we intend to further explore the optimal path of the tillage index, thus making it an endogenous part of the model.

5. Concluding Comments

The biophysical representation of the nutrient cycle in soil degradation models is an important factor in evaluating optimal paths of fertilizer application and the sustainability of agricultural systems. Inclusion of multiple state variables in the form of nutrient pools allows for non-monotonic paths of nutrient stocks and fertilization schedules, which subsequently impact the characterization of the sustainability of the system under alternative criteria. In particular, inclusion of the decadal pool allows for an explicit representation of tillage practices in the model, essential for an analysis of these management decisions. Previous analyses neither addressed these multiple stocks, nor analyzed the sustainability of optimal responses.

While the model presented in this paper incorporates these features, it does have several shortcomings. For example, only interior solutions are considered, essentially ensuring stability of the endogenous value of each nutrient pool over time. In reality, we would expect that fertilizer application would certainly be constrained positive, but also might be constrained from above as a result of regulation or chemical reality. Further analysis in the presence of these constraints is forthcoming in Bond (2004). Furthermore, there has been some evidence that fertilizer and indigenous nutrients are not perfect substitutes, in that long-run fertilizer use adversely affects yield levels (Kim, et. al 2001). This is essentially an empirical question, but such a relationship could certainly be incorporated into our model. We have also constrained the analysis to one limiting nutrient and one choice variable, for simplicity. Allowing for the endogeneity of harvest index and tillage may provide a richer analysis, but at the expense of enormous complication. Additional nutrients would also add complexity, and make the model essentially intractable from an analytical standpoint. Nevertheless, numerical simulation methods could be used to solve the more complicated problems, and numerical analysis of the sustainability criteria and comparative statics are possible.

This general model, which utilizes a biogeochemical structure commonly used in other disciplines, can be used to analyze a wide variety of issues relating to sustainable agriculture. The sustainability criteria developed here, which incorporate both physical and economic notions of sustainability, can also help to shed light on what exactly is to be sustained and over what time scale, often neglected in other scientific literature.

Appendix A: Derivation of $F(N_1, N_2, \lambda_1)$ and Conditions that Guarantee an Interior Solution

Assuming a non-binding constraint and the linear-quadratic production function $Y(N_a) = bN_a - dN_a^{\,2}$, use (8) to solve for F in terms of N_1, N_2, and λ_1, to obtain

$$F(N_1, N_2, \lambda_1) = -\frac{c - \gamma_{atm}(1-\eta)\big(bp + (1-H)\lambda_1 - 2dp(1-\eta)\big(k_2TN_2 + \gamma_3 - k_1(1-\rho_2-\rho_3)N_1\big)\big)}{2dp\gamma_{atm}^{\;2}(1-\eta)^2}.$$

and note that the denominator is positive. In order for the non-binding assumption to hold, it must be the case that $(1-\eta)p\gamma_{atm}\big(b-(1-\eta)2d\gamma_3\big) + (1-\eta)(1-H)\gamma_{atm}\lambda_1 \geq c + (1-\eta)^2 2dp\gamma_{atm}\big(k_1(1-\rho_2-\rho_3)N_1 + k_2TN_2\big)$ for all t in the planning horizon. As λ_1 is constant for an interior solution (see (21)), this implies for non-trivial problems that the stock levels N_1 and N_2 never get "too large" relative to the parameters and steady state level of the shadow value for the active pool. There are four cases to consider (see Figure 2a for further insight). If $(N_{i0} - N_{i\infty})<0$ for i=1,2, then an interior solution is guaranteed due to the monotonicity of the state paths (see Propositions 1 and 2). If $(N_{10} - N_{1\infty})>0$ and $(N_{20} - N_{2\infty})<0$, or both $(N_{i0} - N_{i\infty})>0$ and the condition above is satisfied at time t=0, then F will remain positive through the planning horizon as well (this follows from our assumptions on T, namely that $k_1>k_2T$). Finally, if $(N_{10} - N_{1\infty})<0$ and $(N_{20} - N_{2\infty})>0$, then the initial conditions must be such that if optimal fertilization is positive at time t=0 and the derivative of F with respect to time is less than zero, then the condition above is satisfied over the time period where F is declining.

Appendix B: The Steady State Values

Write the system of equations (14)-(17) in the form $\dot{\mathbf{X}} = \mathbf{JX} + \mathbf{b}$, and note any steady state is defined as the solution to the system when $\dot{\mathbf{X}} = \mathbf{0}$. If the coefficient matrix $\mathbf{J}$ is non-singular for this linear system, then the steady-state exists, is unique, and is defined by $\mathbf{X}_\infty = \mathbf{-J^{-1}b}$. Using the determinant result from Appendix C and carrying out this calculation using the computer program *Mathematica 5.0*, the steady-state solution to the problem is thus

$$\begin{pmatrix} N_{1\infty} \\ N_{2\infty} \\ \lambda_{1\infty} \\ \lambda_{2\infty} \end{pmatrix} = \begin{pmatrix} \dfrac{p(k_1+r)(r+k_2T)\gamma_{atm}(b-bH+2d\gamma_{atm})(1-\eta) + c(1-H)\Big(\big(-(r+k_2T)\big)(r+k_1(H+\eta(1-H)) - (1-H)k_1r(1-\eta)\rho_2 - (1-H)k_1(r+k_2T)(1-\eta)\rho_3\Big)}{2dk_1p(k_1+r)(r+k_2T)\gamma_{atm}(1-\eta)} \\ \dfrac{\rho_2\Big[p(k_1+r)(r+k_2T)\gamma_{atm}(b-bH+2d\gamma_{atm})(1-\eta) + c(1-H)\Big(\big(-(r+k_2T)\big)(r+k_1(H+\eta(1-H)) - (1-H)k_1r(1-\eta)\rho_2 - (1-H)k_1(r+k_2T)(1-\eta)\rho_3\Big)\Big]}{2dk_2Tp(k_1+r)(r+k_2T)\gamma_{atm}(1-\eta)} \\ \dfrac{ck_1\big(k_2T(1-\rho_3) + r(1-\rho_2-\rho_3)\big)}{(k_1+r)(r+k_2T)\gamma_{atm}} \\ \dfrac{ck_2T}{\gamma_{atm}(r+k_2T)} \end{pmatrix}$$

The steady state value F_∞ can be obtained by substitution of these values into the expression for F reported in Appendix A.

Appendix C: The Determinant of J and Dockner (1985) Theorem 3

The determinant of $\mathbf{J} = k_1k_2T(r+k_1)(r+k_2T) > 0$. Theorem 3 of Dockner (1985) gives conditions under which the four eigenvalues of $\mathbf{J}$ are real with two positive and two negative, thus ensuring the saddle point property of the solution to (14)-(17). Assuming K<0, Dockner's equation (20) can be rearranged, as in Tahvonen (1991), so that the condition is $\mathbf{K}^2 - 4\det(\mathbf{J}) \geq 0$, where $\mathbf{K}$ is defined as

$\mathbf{K} = \begin{vmatrix} \partial\dot{N}_1/\partial N_1 & \partial\dot{N}_1/\partial\lambda_1 \\ \partial\dot{\lambda}_1/\partial N_1 & \partial\dot{\lambda}_1/\partial\lambda_1 \end{vmatrix} + \begin{vmatrix} \partial\dot{N}_2/\partial N_2 & \partial\dot{N}_2/\partial N_2 \\ \partial\dot{\lambda}_2/\partial N_2 & \partial\dot{\lambda}_2/\partial\lambda_2 \end{vmatrix} + 2\begin{vmatrix} \partial\dot{N}_1/\partial N_2 & \partial\dot{N}_1/\partial\lambda_2 \\ \partial\dot{\lambda}_1/\partial N_2 & \partial\dot{\lambda}_1/\partial\lambda_2 \end{vmatrix}$. It can be shown that in this case, $\mathbf{K} = -k_1^{\,2} - rk_1 - rk_2T - (k_2T)^2 < 0$, and $\mathbf{K}^2 - 4\det(\mathbf{J}) = (k_1-k_2T)^2(k_1+r+k_2T)^2 > 0$, so that the condition in Theorem 3 is satisfied and the solution to the system is a saddle point.

Appendix D: Derivation of the Explicit Solution

Using *Mathematica 5.0*, we find the negative eigenvalues of **J** are defined as $r_1 = -k_2 T$, $r_2 = -k_1$. Using these values to define the eigenvectors $\mathbf{v}^i$ as those vectors that satisfy $\mathbf{J}\mathbf{v}^i = r_i \mathbf{v}^i$, we find that the two (linearly independent) eigenvectors of the system are $\mathbf{v}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}'$ and $\mathbf{v}^2 = \begin{pmatrix} 1 & \rho_2 k_1 /(-k_1 + k_2 T) & 0 & 0 \end{pmatrix}'$. Substitute these values into the general solution $\boldsymbol{\varphi}(t;\mathbf{x}_\infty, \mathbf{x_0}, \mathbf{A}) = \mathbf{x}_\infty + c_1 \mathbf{v^1} e^{r_1 t} + c_2 \mathbf{v^2} e^{r_2 t}$ to obtain the solutions in terms of the constants c_1 and c_2. Using the initial conditions N_{i0}=0 for i=1,2, the first two equations of the system can be solved at time t=0 to obtain $c_1 = (N_{20} - N_{2\infty}) - \frac{\rho_2 k_1}{k_1 - k_2 T}(N_{1\infty} - N_{10})$ and $c_2 = (N_{10} - N_{1\infty})$. Substitution yields the results reported in (19) - (22).

Figure 1: Schematic of the Nutrient Cycle

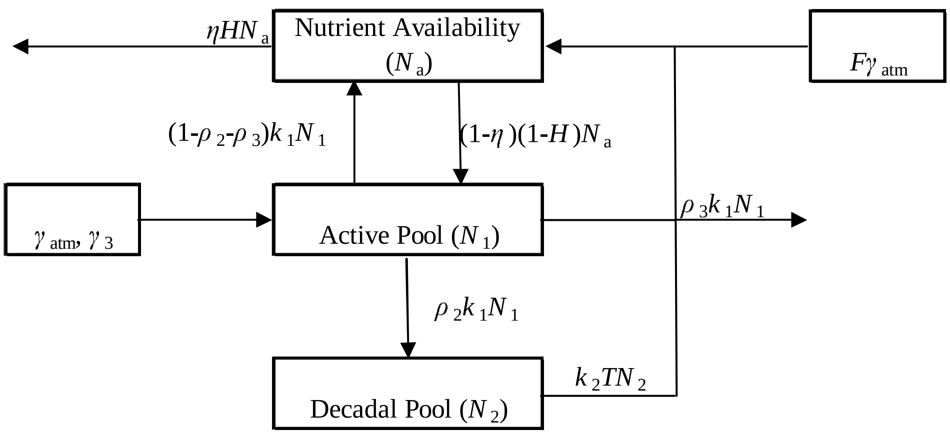
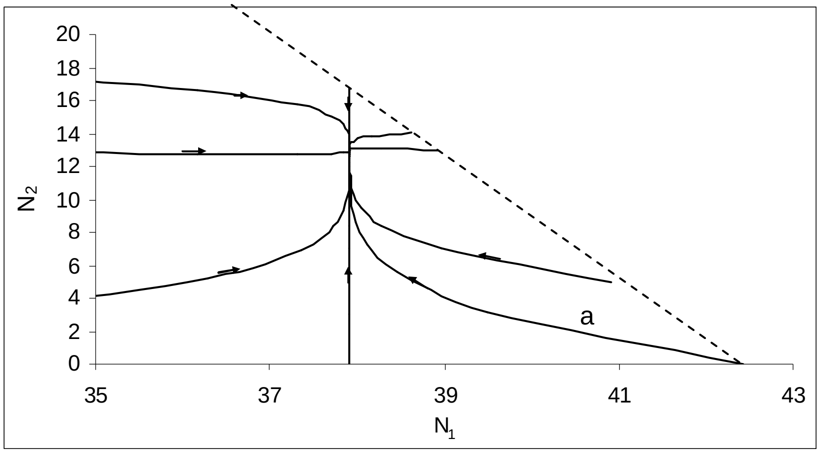
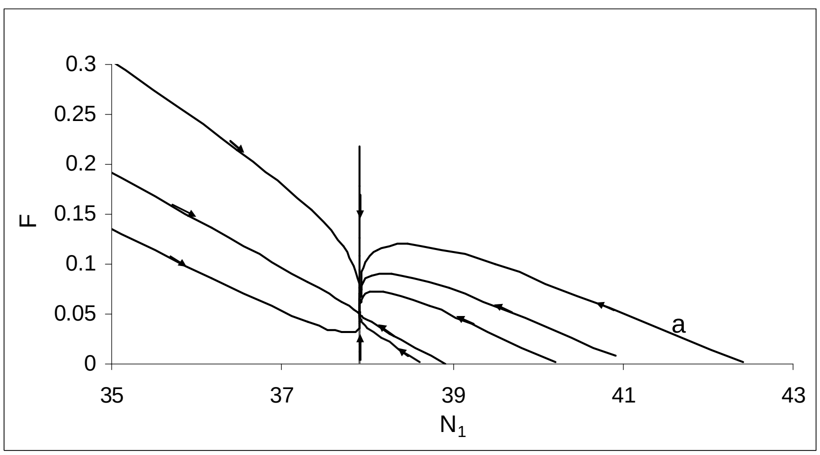


Figure 2: Optimal Trajectories and Non-Negativity Constraint

a)



b)



c)

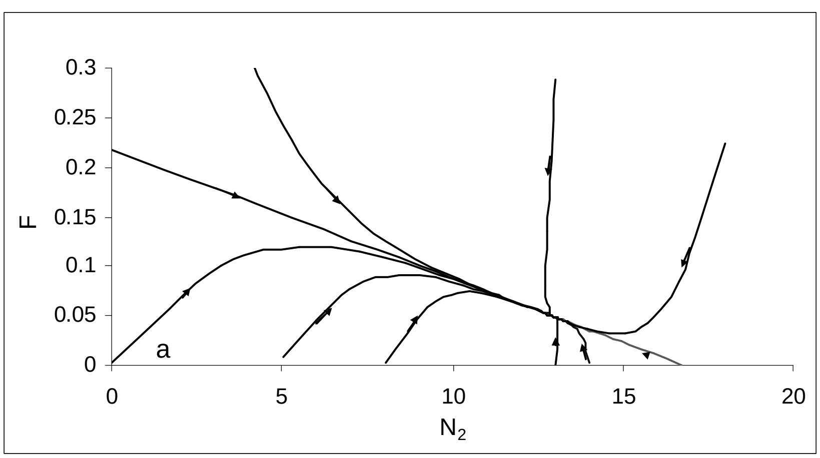
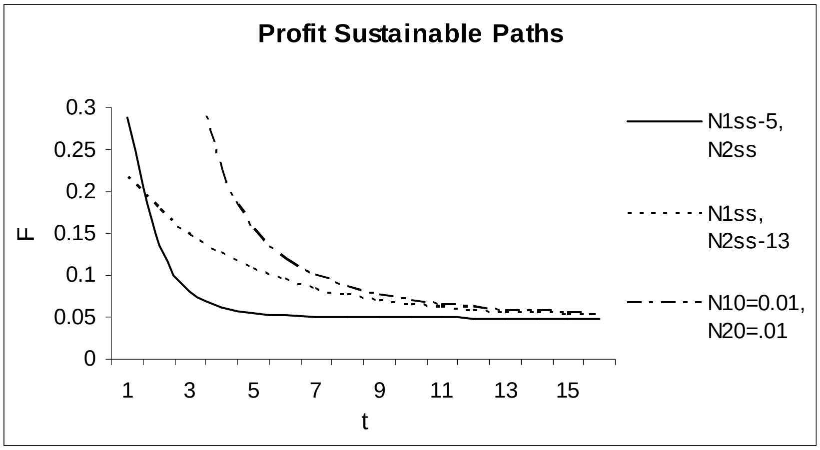


Figure 3: Optimal Fertilization Schedules Over Time

a)



b)

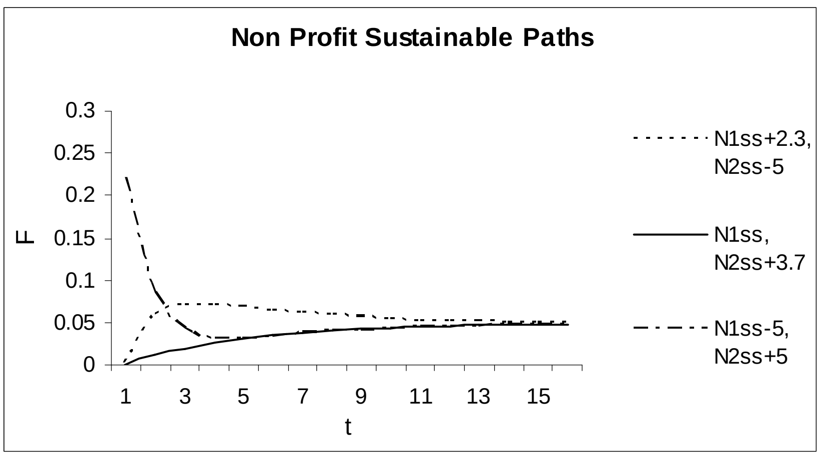


Table 1: Comparative Statics of the Steady State

	Variable				
	$N_{1\infty}$	$N_{2\infty}$	$\lambda_{1\infty}$	$\lambda_{2\infty}$	F_{∞}
p	+	+	0	0	+
c	-	-	+	+	-
r	-	-	-	-	-
b	+	+	0	0	+
d	-	-	0	0	-
T	+	+/-	+	+	+
H	-	-	0	0	-

Notes

[1] The authors argue that due to the chemistry of soils, the model can be written in terms of carbon pools using fixed C:N ratios for each pool. However, in the interests of simplicity, the analysis is performed using N.
[2] Of course, these assumptions represent a restricted case of a more general model that allows for differential export rates for fertilizer and imperfect substitution between fertilizer and indigenous nitrogen. While we recognize this potential for generalization, we maintain that the benefits from the relative simplicity outweigh any potential costs.
[3] In other words, $a_{11} = \left[(1-\eta)(1-H)(1-\rho_2-\rho_3)-1\right]k_1$, $a_{12} = (1-\eta)(1-H)k_2T$, $a_{21} = \rho_2 k_1$, and $a_{22} = -k_2T$.
[4] We keep the solution for F in terms of the state and costate variables due to complexity of the solution.
[5] We take here the parameters in Baisden and Amundson (2003) for their 600x10^3 year old soil: k_1=1.05, k_2=.052, k_3=*.0002,* ρ_2=.085, ρ_3=.0012, η=.061, γ_{atm}=20, T=5, H=0.5, c=.05, p=2, b=0.8, and d=0.01.
[6] There is little evidence in the literature that surplus nutrient availability decreases yields; rather, it generally implies a switching of the limiting nutrient or similar element necessary for crop growth.

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