The Economics of Shallow Lakes[1]

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Abstract

Ecological systems such as shallow lakes are usually non-linear and display discontinuities and hysteresis in their behaviour. These systems often also provide conflicting services as a resource and a waste sink. This implies that the economic analysis of these systems requires to solve a non-standard optimal control problem or, in case of a common property resource, a non-standard differential game. This paper provides the optimal management solution and the open-loop Nash equilibrium for a dynamic economic analysis of the model for a shallow lake. It also investigates whether it is possible to induce optimal management in case of common use of the lake, by means of a tax. Finally, some remarks are made on the feedback Nash equilibrium.

Key words: non-linear differential games, ecological systems.

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1. Introduction

The purpose of this paper is to develop an economic analysis of the shallow lake. Lakes have been studied intensively and the shallow lake model is well tested and documented, so that the analysis has a direct meaning. However, the lake model can also be viewed as a metaphor for many of the ecological problems facing mankind today, so that the analysis developed in this paper will have a wider applicability. The economic analysis is especially challenging because of the non-linear dynamics of the lake (which yields non-convex decision problems) and the gaming aspects related to the common property character of the lake.

It has been observed that shallow lakes, due to a heavy use of fertilizers on surrounding land and an increased inflow of waste water from human settlements and industries, at some point tend to flip from a clear state to a turbid state with a greenish look caused by a dominance of phytoplankton (Carpenter and Cottingham, 1997; Scheffer, 1997). The release of nutrients, especially phosphorus, into the lake stimulates the growth of phytoplankton and in addition to that, the resulting turbidity prevents light to reach the bottom of the lake so that submerged vegetation disappears. With the vegetation many species disappear such as waterfleas which graze on phytoplankton. It has also been observed that shallow lakes are hard to restore in the sense that the nutrient loads have to be reduced below the level where the flip occurred before the lake flips back to a clear state. The positive feedback through the effect on the submerged vegetation is one explanation for this so-called hysteresis effect.

Ecological systems often display discontinuities in the equilibrium states of the system over time. A seminal paper in this area models the sudden outbreak of an insect, called the spruce budworm, and the long time it takes before the budworm density jumps back to a low number again (Ludwig, Jones and Holling, 1978). Technically, this hysteresis effect can be modelled by a non-linear differential equation which has multiple steady-states with separated domains of attraction in a certain range of the exogenous variable. Other examples of ecological systems with hysteresis, among which the lake model, are described in Ludwig, Walker and Holling (1997).

In the ecological literature, management of shallow lakes is mostly interpreted as preventing the lake to flip or, if it flips, as restoring the lake in its original state. However, this approach denies the economics of the problem in the sense of the trade-offs between the utility of the agricultural activities, which are responsible for the release of phosphorus, and the utility of a clear lake. When the lake flips to a green turbid state, the value of the ecological services of the lake (e.g. the intake of water and recreation) decreases but this situation corresponds to a high level of agricultural activities. It depends, of course, on the relative weight attached to these welfare components whether it is better to keep the lake clear or not. Note that if it is better to keep the lake clear, it is very costly to let the lake flip first because of the hysteresis. A second economic issue is that lakes are usually common property resources and therefore suffer from sub-optimal use, in the absence of coordination.

The literature on the lake model is rapidly increasing. Carpenter, Ludwig and Brock (1999) focus on hysteresis and irreversibility issues. The paper that comes closest to this one is by Brock and Starrett (1999). They consider the dynamics and the optimal management of the lake and point out the occurrence of saddle-point stable steady-states and Skiba points. This paper extends their analysis to Nash equilibria, for the game of common property, and to tax policies with the aim to internalize the externalities (see below). Brock and de Zeeuw (2002) consider a repeated game version of the lake model. They show that the occurrence of "bad" Nash equilibria can in fact be beneficial because with these points as threats in trigger strategies, cooperation can be sustained for lower values of the discount factor.

In the first part of the paper, very simple welfare analysis is done on the possible steady-states of the lake model. Relative weights are chosen such that it is optimal to manage the lake in one of its clear states, called oligotrophic states. It is shown, however, that when the lake is shared by more than one community, two Nash equilibria occur: one in an oligotrophic state and one in a dirty state, called a eutrophic state. In the second part of the paper, intertemporal welfare is maximized subject to the dynamics of the lake. It is shown that in case the discount rate is low enough, an optimal path for phosphorus loadings exists, from each initial condition of the lake, which moves the lake towards its optimal steady-state. When the lake is shared by more than one community, a non-linear differential game has to be solved. The phase-diagram for the open-loop Nash equilibrium has three steady-states, two of which are saddle-point stable and correspond to the Nash equilibria found in the first part of the paper. The third point is unstable and displays complex dynamics. However, it is shown that a so-called Skiba point exists which splits the possible initial conditions of the lake in an area from where the Nash equilibrium loading trajectory will approach the oligotrophic saddle-point, and an area from where the eutrophic saddle-point results.

The question arises whether it is possible, by levying a tax on the loading of phosphorus, to induce the communities to follow an optimal management path. Note that if the communities are locked in the eutrophic Nash equilibrium, a straight path to the optimal steady-state is not feasible due to the hysteresis. Assuming that it is not possible to implement a time-varying tax, the answer depends on the number of communities. It is shown that if the number is low enough, a constant tax yields a Nash equilibrium path that moves towards the optimal steady-state (although this path will not be the same as the optimal management path, of course). If the number is high, however, more saddle-points arise again and the dynamics becomes very complex, so that there is no guarantee that a constant tax can induce optimal management of the lake in the long run.

A final issue regards the type of Nash equilibrium employed in the analysis. It is well-known that the open-loop Nash equilibrium is not strongly time-consistent and therefore a feedback Nash equilibrium is preferred. However, due to the non-linear dynamics of the lake, it is very difficult to find a feedback Nash equilibrium. In the last section of the paper, some preliminary remarks are made on this issue. The problem is a one-dimensional infinite horizon differential game, so that the techniques developed by Tsutsui and Mino (1990) for dynamic duopolies with sticky prices, may apply. This would imply the occurrence of multiple equilibria, possibly with welfare levels close to optimal management. The complete analysis is left for further research.

The paper is organized as follows. Section 2 describes the shallow lake model. Section 3 is concerned with the economics of the lake steady-states and section 4 with the dynamic welfare analysis of the lake. Section 4 contains the case of optimal management, the open-loop Nash equilibrium, the effect of taxes and the feedback Nash equilibrium. Section 5 concludes the paper.

2. The Lake Model

Shallow lakes have been studied intensively over the last two decades and it has been shown that the essential dynamics of the eutrophication process can be modelled by the differential equation

$$\dot{P}(t) \;=\; L(t) - sP(t) + r\frac{P^2(t)}{P^2(t) + m^2}, \;\; P(0) = P_0, \tag{1}$$

where P is the amount of phosphorus in algae, L is the input of phosphorus (the "loading"), s is the rate of loss consisting of sedimentation, outflow and sequestration in other biomass, r is the maximum rate of internal loading and m is the anoxic level (see for an extensive treatment of the lake model Carpenter and Cottingham (1997) or Scheffer (1997)). Less is known about deep lakes but from what is known now, it can be expected that the same type of model will be adequate. However, estimates of the parameters of this differential equation for different lakes vary considerably, so that a wide range of possible values has to be considered.

By substituting $x = P/m$, $a = L/r$, $b = sm/r$ and by changing the time scale to rt/m, equation (1) can be rewritten as

$$\dot{x}(t) \;=\; a(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1}, \;\; x(0) = x_0. \tag{2}$$

In order to understand some of the important features of this model, suppose that the loading a is constant. What happens depends on the value of the parameter b. If $b \geq 3\sqrt{3}/8$, all values of a lead to one stable steady-state (see figure 1). If $b \leq ½$, values of a above the local maximum of the curve of steady-states in figure 2 lead to one stable steady-state again. However, values of a below this local maximum yield two stable steady-states for the differential equation (2). The domains of attraction are determined by the unstable steady-state in between: to the right of this point the high stable steady-state results and to the left the low one. If $½ < b < 3\sqrt{3}/8$, values of a below the local minimum and above the local maximum of the curve of steady-states in figure 3 lead to one stable steady-state. For values of a in between two stable steady-states occur again for the differential equation (2), with domains of attraction divided by the unstable steady-state.

It is easy to see a hysteresis effect now for $b < 3 \;\; 3/8$. If the loading a is gradually increased, at first the steady-state level of phosphorus remains low: the lake remains in an oligotrophic state with a high level of ecological services. At a certain point, however, the lake *flips* to a high steady-state level of phosphorus. To put it differently, the lake *flips* to a eutrophic state with a low level of ecological services. If it is then decided to lower the loading a in order to try to bring the lake back to an oligotrophic state, it is not enough to reduce a just below that flip-point. If b is high enough ($½ < b < 3 \;\; 3/8$, figure 3), it can still be done, but a has to be reduced further until the lake flips back to an oligotrophic state. If $b \quad ½$ (figure 2), however, then the lake is trapped in high steady-state levels of phosphorus which means that the first flip is irreversible. In that case, only a change in the parameter b (e.g., by releasing a certain type of fish and thus changing the fauna) can restore the lake. In the sequel of the paper, it is assumed that the parameter $b = 0.6$ so that the lake displays hysteresis but a flip to a eutrophic state is reversible. Furthermore, the loading a will not be exogenous anymore but subject to control. In section 3, a is still constant and the trade-off is considered between the benefits of being able to release that constant amount of phosphorus, on the one hand, and the resulting damage to the lake, on the other hand. Section 4 provides a full dynamic analysis where a can change over time.

[Insert figures 1, 2, 3 about here]

3. Economic Analysis of the Lake Steady-States

Several interest groups operate in relation with the lake, that was modelled in section 2. Because the release of phosphorus into the lake is due to agricultural activity, farmers have an interest in being able to increase the loading. In that way, the agricultural sector can grow without the need, for example, to invest in new technology in order to decrease the emission-output ratio. On the other hand, a clean lake is preferred by fishermen, drinking water companies, other industry that makes use of the water, and people who spend leisure time on or along the lake. In general, the lake is used as a waste sink (for example, by farmers in their activity as multiple non-point source polluters) and as a resource (for example, by water utilities and recreational users). Suppose a community or country, balancing these different interests, can agree on a welfare function of the form $ln\ a - cx^2,\ c > 0$. The lake has value as a waste sink for agriculture ($ln\ a$), for example, and it provides ecological services that decrease with the amount of phosphorus ($-cx^2$). The parameter c reflects the relative weight of these welfare components. Suppose, furthermore, that the lake is shared by n communities or countries with the same welfare function. In this section it is assumed that the communities choose constant loading levels a_i, $i = 1,...,n$, and that the amount of phosphorus adjusts instantaneously to its steady-state level. A logarithmic functional form for the welfare function is chosen, because it is convenient for the technicalities of the analysis and because the optimal management outcome in terms of total loading will be independent of the number of communities. This is helpful because the number of communities can be varied while the optimal management outcome as a benchmark remains the same. It is assumed that the area around the lake is large enough so that adding new communities does not lead to crowding out: the objectives are assumed to be additive in the number n.

Optimal management of the lake requires to solve

$$maximize \sum_{i=1}^{n} \ln a_i - ncx^2 \;\; s.t. \;\; a - bx + \frac{x^2}{x^2+1} = 0, \; a = \sum_{i=1}^{n} a_i. \tag{3}$$

Simple calculus shows that the optimal amount of phosphorus is determined by

$$b - \frac{2x}{(x^2+1)^2} - 2cx(bx - \frac{x^2}{x^2+1}) = 0. \tag{4}$$

Optimal management, of course, does not necessarily yield an oligotrophic state for the lake. If the communities attach a relatively low weight c to ecological services, it can be optimal to choose a eutrophic state with a high level of agricultural activities. It is easy to show that for large values of c, the optimal management problem has one maximum for an x below the flip-point. As the value of c is decreased, first a local maximum appears for a high x whereas the global maximum is still achieved for a low x, but for c low enough (i.e. $c \quad 0.36$) the global maximum occurs for a high x beyond the flip-point. In the sequel of the paper it is assumed that enough weight (i.e. $c = 1$) is attached to the services of the lake to make it optimal to aim for an oligotrophic state.

If $c = 1$, equation (4) yields $x^* = 0.33$ with total loading $a^* = 0.1$. Note that the same level of total loading can also lead to the eutrophic state $x = 1$, if the initial amount of phosphorus is in the upper domain of attraction (see figure 3). A flip occurs when total loading is increased to $a = 0.1021$, so that the lake is managed not far from what is called the "edge of hysteresis" (Brock, Carpenter and Ludwig, 1997). A small mistake may cause a flip with high costs, not only directly because of a jump to a high x but also indirectly because of the long return path.

If the communities do not cooperate in managing the lake, it is assumed a Nash equilibrium results which requires to solve

$$\textit{maximize}\ \ln a_i - cx^2,\ i = 1,\ldots,n,\ \textit{s.t.}\ a - bx + \frac{x^2}{x^2+1} = 0,\ a = \sum_{i=1}^{n} a_i. \tag{5}$$

Simple calculus shows that the Nash equilibrium level of phosphorus is determined by

$$b - \frac{2x}{(x^2+1)^2} - 2\frac{c}{n}x\left(bx - \frac{x^2}{x^2+1}\right) = 0. \tag{6}$$

If $c = 1$ again and if the number of communities $n = 2$, equation (6) has three solutions, two of which correspond to a Nash equilibrium. The first Nash equilibrium yields $x^N{}_1 = 0.36$ with total loading $a^N{}_1 = 0.1012$. The lake is still in an oligotrophic state but closer to the edge of hysteresis. However, the second Nash equilibrium yields an eutrophic state $x^N{}_2 = 1.51$ with total loading $a^N{}_2 = 0.2108$. Welfare under optimal management and in the oligotrophic Nash equilibrium are comparable, but welfare in the eutrophic Nash equilibrium is much lower. Moreover, when the communities are locked into the second Nash equilibrium and decide to coordinate, it is much more difficult to reach the optimal management outcome, due to the hysteresis. It is not enough to reduce total loading to 0.1. It has to be reduced to 0.0898 first, in order to flip back to an oligotrophic state, and can then be increased to 0.1 again.

If $n > 2$, these numbers change of course, but it is easy to see that for all b in the range with hysteresis and reversibility ($\tfrac{1}{2} < b < 3\ \ 3/8$), on which this paper focuses, always two Nash equilibria occur. In fact, equation (6) intersects the curve for the lake steady-states with the curve described by $(n/2cx)(b - 2x/(x^2 + 1)^2)$. For b in the range given above, this curve has a negative part for x in a positive range. Furthermore, it approaches infinity for x 0 and it approaches zero from above for x . Increasing the number of communities n implies that the curve is stretched out but the three intersection points remain, two of which are Nash equilibria.

In the next section the loading a can change over time and the amount of phosphorus does not adjust instantaneously to its steady-state level but gradually according to equation (2), which turns the optimal management problem into an optimal control problem and the static game into a differential game. The Nash solutions found in this section return (approximately) as saddle-point stable steady-states with solution trajectories that may have to bend around the flip-point.

4. Dynamic Economic Analysis of the Lake Model

Suppose that the problem has an infinite horizon, so that the objectives become

$$W_i = \int_0^\infty e^{-\rho t}[\ln a_i(t) - cx^2(t)]\,dt, \;\; i = 1,\ldots,n, \tag{7}$$

where *ñ > 0* is the discount rate.

4.1 Optimal management

Optimal management requires to maximize the sum of the objectives W_i, subject to equation (2) with $a = Óa_i$. This is an optimal control problem and the maximum principle yields the necessary conditions

$$\frac{1}{a_i(t)} + \lambda(t) = 0, \;\; i = 1,\ldots,n, \tag{8}$$

$$\dot{\lambda}(t) = [(b+\rho) - \frac{2x(t)}{(x^2(t)+1)^2}]\lambda(t) + 2ncx(t), \tag{9}$$

with a transversality condition on the co-state ë, and equation (2). Using (8), equation (9) can be rewritten as a set of identical differential equations in a_i, $i = 1,\ldots,n$. The sum of these equations (or multiplication of one of them by *n)* yields a differential equation in total loading *a:*

$$\dot{a}(t) = -[(b + \rho) - \frac{2x(t)}{(x^2(t) + 1)^2}]a(t) + 2cx(t)a^2(t). \tag{10}$$

The solution is given by the set of differential equations (2) and (10), and the transversality condition. Note that $b = 0.6$ (see section 2) and $c = 1$ (see section 3). The phase-diagram in the (x, a)-plane is drawn in figure 4a. One curve represents the steady-states for x and can be recognized as the lake steady-states, which were discussed in sections 2 and 3. The other curve represents the steady-states for a. Its position depends on the discount rate $\tilde{n}$. If the discount rate is low enough ($\tilde{n} < 0.1$), this curve intersects the first curve only once in a point that is saddle-point stable. If the discount rate is higher, the second curve moves up, it intersects the first curve three times, and the analysis becomes similar to the analysis of the open-loop Nash equilibrium below. It is assumed here that the discount rate $\tilde{n} = 0.03$, which yields the graph in figure 4a. The steady-state is close to the static optimal management solution in section 3, and converges to that point when the discount rate goes to 0. The optimal solution prescribes to jump, at any initial state of the lake, to the stable manifold and to move towards the steady-state. Given the non-linearity of the problem, it is not easy to obtain an analytical expression for the stable manifold but a numerical approximation is not difficult to develop. Starting at the steady-state point, the characteristic vector corresponding to the negative eigenvalue of the Jacobian matrix determines the direction of the stable manifold. Working backwards from the steady-state in small steps, a piecewise linear approximation of the stable manifold is then found and the approximation gets better the smaller the steps. With Mathematica (Wolfram, 1999), the stable and unstable manifolds for the set of differential equations (2) and (10) can be drawn (see figure 4b). Note that the stable manifold can be reached from all initial states x_0 and bends around the lower flip-point (see also section 3).

[Insert figures 4a, 4b about here]

4.2 Open-loop Nash equilibrium

The open-loop Nash equilibrium (Başar and Olsder, 1982) is found by applying the maximum principle to each objective W_i, $i = 1,...,n$, separately (fixing a_j for $j \neq i$) subject to equation (2) with $a = \Sigma a_i$. The set of necessary conditions becomes

$$\frac{1}{a_i(t)} + \lambda_i(t) \;=\; 0,\;\; i = 1,...,n, \tag{11}$$

$$\lambda_i(t) \;=\; [(b+\rho) - \frac{2x(t)}{(x^2(t)+1)^2}]\lambda_i(t) + 2cx(t),\;\; i = 1,...,n, \tag{12}$$

with transversality conditions on the co-states λ_i, and equation (2). Using (11), equations (12) can be rewritten as differential equations in a_i, $i = 1,...,n$. These equations are identical and the sum (or multiplication of one of them by n) yields a differential equation in total loading a:

$$\dot{a}(t) \;=\; -[(b+\rho) - \frac{2x(t)}{(x^2(t)+1)^2}]a(t) + 2\frac{c}{n}x(t)a^2(t). \tag{13}$$

The open-loop Nash equilibrium is given by the set of differential equations (2) and (13), and the transversality conditions. The phase-diagram for two communities $n = 2$ (and $b = 0.6$, $c = 1$, $\rho = 0.03$) in the (x, a)-plane is drawn in figure 5a. The steady-state curves for x and a now have three intersection points. The intersection points on the left and on the right are saddle-point stable and yield possible steady-states for the Nash equilibrium in an oligotrophic and in a eutrophic area, respectively. The intersection point in the middle is unstable with complex eigenvalues. Again with Mathematica (Wolfram, 1999), the stable and unstable manifolds for the set of differential equations (2) and (13) can be drawn (see figure 5b). The trajectories of the stable manifold curl a while from the intersection point in the middle and then go either to the steady-state on the left or to the steady-state on the right. It is clear that when the initial state x_0 lies to the right of the set of curls, the open-loop Nash equilibrium follows the upper trajectory to the steady-state on the right, and when the initial state lies to the left of that area, it follows the lower trajectory to the steady-state on the left. However, it is more difficult to see what happens in the range in between. It can be shown (Appendix A) that a state x_S exists such that for $x_0 < x_S$, the open-loop Nash equilibrium jumps to the lower trajectory and moves towards the oligotrophic steady-state whereas for $x_0 > x_S$, it jumps to the upper trajectory and moves towards the eutrophic steady-state. The point x_S is called a Skiba point because it was introduced by Skiba in an optimal growth model with a convex-concave production function (Skiba, 1978, Brock and Malliaris, 1989).

[Insert figures 5a, 5b about here]

If *n > 2*, the same arguments as in section 3 can be used to show that always two open-loop Nash equilibria occur. Note, however, by inspection of equation (13), that the arguments do not hold for all *b* in the range *(½, 3   3/8)* anymore, because of the positive discount rate *ñ*, but only for *b + ñ < 3   3/8*, which holds for the specific values chosen for *b* and *ñ*.

Before turning to the question whether a tax can induce the communities to choose loadings according to the optimal management trajectory, it is useful to make a few general remarks on the analysis above. First, when comparing equations (10) and (13), it is immediately clear that the open-loop Nash equilibrium also results under optimal management with parameter *c/n* instead of parameter *c*. It is an example of a potential game where Nash equilibria can be found by maximizing some adapted objective (Monderer and Shapley, 1996; Dechert and Brock, 1999). Second, it also means that all outcomes considered here (optimal management with varying relative weight *c*, and symmetric open-loop Nash equilibria with fixed *c* but varying number of communities *n*) can be traced by solving an optimal control problem where the set of differential equations, characterizing the solution, has a parameter *c/n*. This parameter can be denoted as the bifurcation parameter. It can be shown that in this Hamiltonian system with positive discounting only saddle-node and heteroclinic bifurcations can occur (Wagener, 1999; Brock and Starrett, 1999). Third, figure 5b contains similar dynamics as found in a model for external economies with multiple steady-states (Krugman, 1991; Matsuyama, 1991). Krugman (1991) argues that for initial conditions (history) to the left and to the right of the set of spirals, the economy moves to the left or to the right, respectively, but that for history within the set of spirals, expectations determine where the economy will end up. This is not an answer to uncertainty but since for these initial conditions either a path can be chosen that goes to the left or one that goes to the right, something must determine where the economy will end up. In our model, however, the solution is driven by an objective. A full analysis of the value function shows that the initial conditions determine the outcome, so that only history matters. A Skiba point exists that divides the area of initial conditions into an area that is attracted by the steady-state on the left and an area that is attracted by the steady-state on the right.

4.3 Taxes

Consider the case of achieving the unique steady-state amount of phosphorus under optimal management by a tax ô on phosphorus loading. Under the tax scheme the objectives (7) of the open-loop differential game change to

$$W_i = \int_0^{\infty} e^{-\rho t}[\ln a_i(t) - \tau(t)a_i(t) - cx^2(t)]\,dt, \; i = 1,\ldots,n. \tag{14}$$

The maximum principle requires for the optimal choice of phosphorus loading at each point in time that

$$\frac{1}{a_i(t)} - \tau(t) + \lambda_i(t) = 0, \; i = 1,\ldots,n. \tag{15}$$

In order to obtain the loading that corresponds to optimal management, it is immediately clear by comparing (8) to (15) that the tax on loading should be chosen such that *ô(t) = -ë(t) +* $ë_i(t)$. This implies that the tax bridges the gap between the social shadow cost of the accumulated phosphorus *ë(t)* and the private shadow cost of the accumulated phosphorus $ë_i(t)$ that causes the steady-state phosphorus levels in the open-loop Nash equilibrium to exceed the (unique) steady-state phosphorus level under optimal management. The tax rate, however, is time-dependent, since it has to equalize cooperative and non-cooperative loading at every point in time. Although optimal, such a tax will be very difficult to implement, since it would require a regulating institution to continuously change the tax rate. Another, more realistic, approach would be to choose a fixed tax rate on loading, defined such that the non-cooperative steady-state phosphorus level under the constant tax equals the steady-state phosphorus level under optimal management. This tax will be called the optimal steady-state tax (OSST).

By comparing (9) to (16) and using (10), it is easy to see that the OSST ô* is given by

$$\tau^* = -\frac{(n-1)\lambda^*}{n} = \frac{(n-1)}{a^*}, \tag{16}$$

where ë* is the value of the co-state and a^* is total loading in the steady-state under optimal management. Under this constant tax scheme, the open-loop Nash equilibrium will be given by the set of differential equations (2) and, instead of (13),

$$\dot{a}(t) = -[(b+\rho) - \frac{2x(t)}{(x^2(t)+1)^2}][a(t) - \frac{\tau^*}{n}a^2(t)] + 2\frac{c}{n}x(t)a^2(t) \tag{17}$$

with a transversality condition.

It is easy to check that the steady-state (x^*, a^*) for the set of differential equations (2) and (10) under optimal management is also a steady-state for the set of differential equations (2) and (17) in the open-loop Nash equilibrium under the constant tax ô*. By substituting $a(t) = a^*$ and ô* = $(n - 1)/a^*$ in the second term between brackets of the right-hand side of equation (17), this term reduces to a^*/n. It is then easy to see that (x^*, a^*) is also a point on the curve representing the steady-states for total loading a in the open-loop Nash equilibrium under the OSST. However, the rest of this curve differs from the one under optimal management.

It should be made clear that the OSST leads to the optimal management steady-state but the path under the OSST that determines the transition to the steady-state *is not the same* as the optimal management path. Coincidence of the optimal management path and the regulated path requires to use the time-dependent tax. To put it differently, the stable manifold of the optimal management problem is not the same as the stable manifold of the regulated problem, although both approach the same saddle-point. Note also that the time of convergence to the steady-state under the OSST will be different than under a time-dependent tax or another control scheme.

If the number of communities $n = 2$, the phase-diagram under the OSST in the (x, a)-plane is drawn in figure 6a. Although this figure differs from figure 4a for optimal management, it is qualitatively the same. It has one saddle-point and a corresponding stable manifold. Starting at both unregulated Nash equilibrium steady-states, the two communities will change their loadings under the OSST and the equilibrium path will follow this stable manifold and move towards the optimal management steady-state. Starting at the oligotrophic Nash equilibrium, this is a short trajectory, but starting at the eutrophic Nash equilibrium, the path has to bend around the flip-point.

[Insert figures 6a, 6b, 6c about here]

Increasing the number of communities n, at a certain point the phase-diagram under the OSST becomes very complicated. From equation (17), the curve representing the steady-states for a is given by

$$a = \frac{(b+\rho) - \dfrac{2x}{(x^2+1)^2}}{2\dfrac{c}{n}x + [(b+\rho) - \dfrac{2x}{(x^2+1)^2}]\dfrac{(n-1)}{na^*}}. \tag{18}$$

The denominator of the right-hand side of equation (18) is zero if and only if

$$\frac{2cx}{(n-1)} + [(b+\rho) - \frac{2x}{(x^2+1)^2}]\frac{1}{a^*} = 0. \tag{19}$$

For $n = 2$ the left-hand side of equation (19) is positive, but for $n > 7$ equation (19) has two roots which yield two vertical asymptotes for the curve given by equation (18). Note that this phenomenon occurs because the term between brackets is partly negative which is caused by the specific choice of b and ñ (see also section 4.2). Moreover, if n goes to infinity, the curve approaches $a = a^*$, but this convergence is not uniform, due to the two discontinuities. An example of such a phase-diagram under the OSST is drawn in figure 6b where $n = 10$. This case still has only one saddle-point stable steady-state. If n gets large, it is to be expected that the curve has more intersection points with the curve representing the lake steady-states, because of the convergence to a^*. This implies the possibility that multiple steady-states occur under the OSST. An example is drawn in figure 6c where $n = 100$. This case has three steady-states again, two of which are saddle-point stable, whereas the middle one is unstable. For a lower discount rate ñ and a higher number of communities n, it may happen that two more steady-states occur between the asymptotes, one unstable and one saddle-point. The existence of a second steady-state characterized by saddle-point stability leads to the conclusion that if the number of communities n is high, the optimal steady-state tax may not work. Depending on the initial conditions, the OSST may direct the equilibrium path towards a steady-state with a higher phosphorus level than in the optimal management steady-state.

4.4 Feedback Nash equilibrium

The open-loop Nash equilibrium is weakly time-consistent but not strongly time-consistent which implies that the equilibrium is not robust against unexpected changes in the state of the lake (Başar, 1989). To obtain an equilibrium with the Markov perfect property, the feedback Nash equilibrium has to be found which means that the Hamilton-Jacobi-Bellman equation for the game has to be solved. This is a difficult problem because it is not clear what the value function will look like due to the complexity of the lake model.

In a linear-quadratic framework with quadratic value functions, the solution would be analytically tractable. In a problem like the one under consideration, one would expect that the steady-state amount of phosphorus and total loadings will be higher in the feedback Nash equilibrium than in the open-loop Nash equilibrium. This would confirm the intuition derived in similar type of problems (van der Ploeg and de Zeeuw, 1992). If a community knows that the other communities will respond to a higher amount of phosphorus in the lake with lower loadings, it loads more at the margin because loading will be partly offset by the reactions of the other communities. Since all communities argue in this way, total loading in the feedback equilibrium is higher than in case the loadings are not conditioned on the state of the lake as in the open-loop Nash equilibrium. However, Tsutsui and Mino (1990) have shown, for a dynamic duopoly model with sticky prices, that non-quadratic value functions exist that solve the Hamilton-Jacobi-Bellman equation. From this it follows that multiple feedback Nash equilibria exist. The resulting set of steady-state prices lies near to the price under full cooperation or collusion. It implies that feedback Nash equilibria exist with steady-state prices that are better for the duopoly than the steady-state price in the open-loop Nash equilibrium. This technique was applied to the international pollution control model by Dockner and Long (1993) to show that (non-linear) feedback equilibria exist that yield lower steady-state stocks of pollution than the open-loop equilibrium. Since the characteristics of those problems (infinite horizon and a one-dimensional state) are the same as for the lake problem, one may expect that the same technique can be applied here. In this section a few steps will be taken but a full analysis is left for further research.

Suppose that the loading strategies in the (symmetric) feedback Nash equilibrium are given by $a_i = h(x),\ i = 1,...,n$. The Hamilton-Jacobi-Bellman equation (or dynamic programming equation) for community $i,\ i = 1,..,n$, becomes

$$\rho V(x) \ = \ \max\,[\ln a_i - cx^2 + V_x(x)[a_i + (n-1)h(x) - bx + \frac{x^2}{x^2+1}]], \tag{20}$$

where V denotes the value function, that is the same for each community.
The first-order condition yields

$$\frac{1}{a_i} \ = \ -V_x(x) \ \Rightarrow \ a_i := h(x) \ = \ \frac{-1}{V_x(x)}. \tag{21}$$

Substitution of (21) into (20) leads to

$$\rho V(x) \ = \ \ln h(x) - cx^2 - \frac{1}{h(x)}[nh(x) - bx + \frac{x^2}{x^2+1}], \tag{22}$$

and differentiation of (22) with respect to x, using (21) again, leads to

$$-\frac{\rho}{h(x)} \ = \ \frac{h'(x)}{h(x)} - 2cx + \frac{h'(x)}{h^2(x)}[-bx + \frac{x^2}{x^2+1}] + \frac{1}{h(x)}[b - \frac{2x}{(x^2+1)^2}]. \tag{23}$$

Rewriting (23) yields an ordinary differential equation in the feedback loading $h(x)$:

$$h'(x)[h(x) - bx + \frac{x^2}{x^2+1}] + h(x)[(\rho + b) - 2cxh(x) - \frac{2x}{(x^2+1)^2}] \ = \ 0. \tag{24}$$

If the resulting steady-state x^F were known, equation (2) with total loading $a = nh(x)$ gives a boundary condition for the differential equation (24):

$$h(x^F) \ = \ \frac{1}{n}[bx^F - \frac{x^{F2}}{x^{F2}+1}], \tag{25}$$

so that the differential equation can be solved. However, this steady-state is a degree of freedom. This means that it is to be expected that multiple feedback Nash equilibria exist. If that steady-state is chosen to be equal to the steady-state under optimal management, it may yield a feedback Nash equilibrium that sustains the optimal management outcome in steady-state. Note that this does not imply that the same trajectory to the steady-state is followed: welfare will generally still be different. However, it is to be expected that a feedback Nash equilibrium exists that is better than the open-loop one. How can this result be reconciled with the intuition described above and the conclusion that the feedback Nash equilibrium is worse than the open-loop one? Note that the analysis above was restricted to quadratic value functions in a linear-quadratic framework so that linear controls result. It seems that by enlarging the strategy spaces to non-linear controls (with non-quadratic value functions), equilibria arise that are better. This may be recognized as a type of folk-theorem in differential games. Further research is needed to be able to give full answers to these issues.

5. Conclusion

Economics of ecological systems is a much neglected area in the literature. Furthermore, the complex dynamics of these systems and the common property aspect of the ecological services as resource and waste sink, present interesting challenges to economic theory. This paper focuses on the shallow lake, as an example but also because much is known about shallow lakes in the ecological literature. However, the analysis in this paper applies to all models that are driven by convex-concave relations, and models with this feature are very typical for mathematical models in ecology (see Murray, 1989).

Internal loading of phosphorus in shallow lakes causes the lake model to be non-linear, with hysteresis effects in the more interesting cases. As a consequence, even if optimal management of the lake has only one steady-state with saddle-point stability, either an increase in the discount rate or an increase in the number of communities, sharing the lake, leads to more saddle-points and complicated dynamics in between. However, a Skiba point exists, which means that in these cases the initial level of accumulated phosphorus determines whether the lake will end up in a clear or a turbid state. For a small number of communities, a constant tax on the loading of phosphorus can induce optimal behaviour and a return to a clear state, but for a large number this policy may not work. The analysis employs the open-loop Nash equilibrium to characterize non-cooperative behaviour. The feedback Nash equilibrium would be more appropriate but is very difficult to identify for these type of problems. Some first steps are given but a full analysis is left for further research.

References

Başar, T. and G.J. Olsder (1982), *Dynamic Noncooperative Game Theory*, Academic Press, New York.

Başar, T. (1989), Time consistency and robustness on equilibria in noncooperative dynamic games, in F. van der Ploeg and A.J. de Zeeuw (eds.), *Dynamic Policy Games in Economics*, Contributions to Economic Analysis 181, North Holland, Amsterdam, 9-54.

Brock, W.A., S.R. Carpenter and D. Ludwig (1997), Notes on optimal management of a lake subject to flips, *Working Paper*, University of Wisconsin.

Brock, W.A. and D. Starrett (1999), Nonconvexities in ecological management problems, *Working Paper*, University of Wisconsin.

Brock, W.A. and A.G. Malliaris (1989), *Differential Equations, Stability and Chaos in Dynamic Economics*, North Holland, Amsterdam.

Brock, W.A. and A. de Zeeuw (2002), The repeated lake game, *Economics Letters*, 76, 109-114.

Carpenter, S.R. and K.L. Cottingham (1997), Resilience and restoration of lakes, *Conservation Ecology* 1:2.

Carpenter, S.R., D. Ludwig and W.A. Brock (1999), Management of eutrophication for lakes subject to potentially irreversible change, *Ecological Applications*, 9(3), 751-771.

Dechert, W.D. and W.A. Brock (1999), Lakegame, *Working Paper*, University of Houston.

Dockner, E.J. and N.V. Long (1993), International pollution control: cooperative versus noncooperative strategies, *Journal of Environmental Economics and Management*, 24, 13-29.

Krugman, P. (1991), History versus expectations, *Quarterly Journal of Economics*, CVI, 651-667.

Ludwig, D., D.D. Jones and C.S. Holling (1978), Qualitative analysis of insect outbreak systems: the spruce budworm and forest, *Journal of Animal Ecology* 47, 315-332.

Ludwig, D., B.H. Walker and C.S. Holling (1997), Sustainability, stability and resilience, *Conservation Ecology* 1:7.

Matsuyama, K. (1991), Increasing returns, industrialization, and indeterminacy of equilibrium, *Quarterly Journal of Economics* CVI, 617-650.

Monderer, D. and L.S. Shapley (1996), Potential games, *Games and Economic Behaviour* 14, 124-143.

Murray, J.D. (1989), *Mathematical Biology*, Springer Verlag, Berlin.

van der Ploeg, F. and A.J. de Zeeuw (1992), International aspects of pollution control, *Environmental & Resource Economics* 2, 117-139.

Scheffer, M. (1997), *Ecology of Shallow Lakes*, Chapman and Hall, New York.

Skiba, A.K. (1978), Optimal growth with a convex-concave production function, *Econometrica* 46, 3, 527-539.

Tsutsui, S. and K. Mino (1990), Nonlinear strategies in dynamic duopolistic competition with sticky prices, *Journal of Economic Theory* 52, 136-161.

Wagener, F.O.O. (1999), Shallow lakes, *Working Paper*, University of Amsterdam.

Wolfram, S. (1999), *The Mathematica Book*, 4th ed., Wolfram Media / Cambridge University Press, Cambridge.

Appendix A: The Skiba point

As also noted later in the main text, the open-loop Nash equilibrium results from maximizing the welfare objective (a potential function)

$$W = \int_0^\infty e^{-\rho t}\Big[\sum_{i=1}^{n} \ln a_i(t) - cx^2(t)\Big]\,dt. \tag{A1}$$

What follows is strongly based on Wagener (1999).
Define the Hamiltonian function

$$H = e^{-\rho t} g(x,a_1,..,a_n) + \mu f(x,a_1,..,a_n), \tag{A2}$$

where

$$g(x,a_1,..,a_n) = \sum_{i=1}^{n} \ln a_i - cx^2,\ f(x,a_1,..,a_n) = \sum_{i=1}^{n} a_i - bx + \frac{x^2}{x^2+1}, \tag{A3}$$

and define the current value Hamiltonian function

$$\tilde{H} = g(x,a_1,..,a_n) + \lambda f(x,a_1,..,a_n),\ \lambda = e^{\rho t}\mu. \tag{A4}$$

The maximum principle yields the necessary conditions (8), (9) and (2) with the parameter c replaced by c/n, which then yields the set of differential equations (2) and (13) for the open-loop Nash equilibrium with the phase-diagram given in figure 5a and the stable and unstable manifolds in figure 5b.

Denote the x-coordinate of the oligotrophic steady-state as x_1 and of the eutrophic steady-state as x_4, and denote the range with curls as $[x_2, x_3]$, where x_2 and x_3 are the x-coordinates of the intersection points of the outer upper curl and the outer lower curl with the curve representing the steady-states for x ($f = 0$), respectively.

Along trajectories, it holds that

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial \mu}\frac{d\mu}{dt} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial \mu} - \frac{\partial H}{\partial \mu}\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} = -\rho e^{-\rho t}g. \quad \text{(A5)}$$

It follows that

$$\tilde{H}(0) = H(0) = -\int_0^\infty \frac{dH}{dt}dt = \int_0^\infty \rho e^{-\rho t} g\, dt = \rho W. \quad \text{(A6)}$$

Furthermore,

$$\frac{dW}{dx} = \frac{dW}{dt}\frac{dt}{dx} = \frac{1}{\rho}\frac{d\tilde{H}}{dt}\frac{1}{f} = \frac{1}{\rho}[\frac{\partial \tilde{H}}{\partial x}f + f(\rho\lambda - \frac{\partial \tilde{H}}{\partial x}]\frac{1}{f} = \lambda. \quad \text{(A7)}$$

Condition (8) yields

$$\lambda = -\frac{1}{a_i},\ i = 1,..,n,\ \Rightarrow\ \lambda = -\frac{n}{a}\ (a = \sum_{i=1}^{n} a_i). \quad \text{(A8)}$$

The proof of the existence of a unique Skiba point takes four steps.

1) Suppose the initial condition x_0 is in the interior of the range $[x_2, x_3]$.

It is better to jump immediately to the upper trajectory instead of to the *same* trajectory some point earlier, or to the lower trajectory instead of to the *same* trajectory some point earlier, because the welfare difference

$$\int \frac{dW}{dx}dx = \int \lambda dx = \int -\frac{n}{a}dx < 0. \quad \text{(A9)}$$

2) Suppose the initial condition is x_2. The choice is either to jump to the upper trajectory and start

at the intersection point with $f = 0$ or to jump to the lower trajectory, by a proper choice of intial loadings a. Because

$$\frac{\partial W}{\partial a} = \frac{1}{\rho}\frac{\partial \tilde{H}}{\partial a} = \frac{1}{\rho}\sum_{i=1}^{n}\frac{\partial}{\partial a_i}(g + \lambda f) = \frac{1}{\rho}\sum_{i=1}^{n}\frac{1}{a_i^2}f = \frac{1}{\rho}\frac{n^3}{a^2}f \quad \text{(A10)}$$

and $f < 0$ below the intersection point, the welfare difference between the upper trajectory and the lower trajectory is negative, so that it is better to jump to the lower trajectory at x_2.

3) Suppose the initial condition is x_3. The choice is either to jump to the lower trajectory and start at the intersection point with $f = 0$ or to jump to the upper trajectory. Because above the intersection point $f > 0$, it follows from equation (A10) that the welfare difference between the upper trajectory and the lower trajectory is positive, so that it is better to jump to the upper trajectory at x_3.

4) Compare now the upper trajectory leading to steady-state on the right, with co-state $\ddot{e}_2$, and the lower trajectory leading to the steady-state on the left, with co-state $\ddot{e}_1$. Denote the welfare difference as *ÄW*. Using the results in steps 2-3 and equations (A7)-(A8), it follows that

$$\Delta W(x_2) < 0,\ \Delta W(x_3) > 0,\ \frac{d}{dx}\Delta W = \lambda_2 - \lambda_1 > 0. \quad \text{(A11)}$$

From (A11) it follows that a point x_S in the interior of the range $[x_2, x_3]$ exists, such that

$$\Delta W(x_S) = 0;\ \Delta W(x) < 0,\ x \in [x_2, x_S);\ \Delta W(x) > 0,\ x \in (x_S, x_3]. \quad \text{(A12)}$$

The point x_S is called a Skiba point and (A12) implies that if the initial amount of phosphorus x_0 is on the left-hand side of the Skiba point, the equilibrium jumps to the lower trajectory and moves towards the steady-state on the left and if the initial amount of phosphorus x_0 is on the right-hand side of the Skiba point, the equilibrium jumps to the upper trajectory and moves towards the steady-state on the right.