The Economics of
Non-Convex Ecological Systems

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Lecture 1: (Non-Convexity in) Optimal Control and Differential Games

Lecture 2: The Economics of Shallow Lakes

• Motivation:

- pollution/resource stocks: optimal control

- common property resource: (differential) game

- non-convexities in ecological systems

(multiple steady states, bifurcations)

(Non-Convexity in)
Optimal Control and Differential Games

- **First: linear-quadratic set-up**

- **Example: International Pollution Control**

- **Pollution by-product of production:** $P = \forall Y$

- N **countries, indexed** i, j

- **Stock pollution:**

$$\dot{S}(t) = \frac{\alpha}{N}\sum_{j=1}^{N} Y_j(t) - \delta S(t), S(0) = S_0$$

- **Welfare indicators:**

$$W_i = \int_0^{\infty} e^{-rt}[\beta Y_i(t) - 0.5Y_i^2(t) - 0.5\gamma S^2(t)]dt$$

• **Optimal control (*i* = *1*); current-value Hamiltonian:**

$$H = \beta Y - 0.5Y^2 - 0.5\gamma S^2 + \lambda(\alpha Y - \delta S)$$

• **Necessary conditions:**

$$\beta - Y + \alpha\lambda = 0$$

$$\dot{\lambda}(t) - r\lambda(t) = \gamma S(t) + \delta\lambda(t)$$

• **Steady states for 8: $(r + *)8 + (S = 0$ (line)**

• **Steady states for S: $\forall\exists + \forall^2 8 - *S = 0$ (line)**

• **Phase diagram in $(S, 8)$ - plane; stable manifold; transversality conditions**

• **Steady state:**

$$S = \frac{\alpha\beta(r+\delta)}{\delta(r+\delta)+\alpha^2\gamma}$$

Benefits of International Coordination

- **Optimal control with** EW_i **:**

$$H = \sum_{i=1}^{N}(\beta Y_i - 0.5Y_i^2) - 0.5\gamma NS^2 + \lambda(\frac{\alpha}{N}\sum_{i=1}^{N}Y_i - \delta S)$$

- **Necessary conditions:**

$$\beta - Y_i + \frac{\alpha}{N}\lambda = 0, i = 1,2,...,N$$

$$\dot{\lambda}(t) - r\lambda(t) = \gamma NS(t) + \delta\lambda(t)$$

- **Steady states for** *8*: ***(r + *)8 + (NS = 0***

- **Steady states for** S: ***∀∃ + (∀²/N)8 - *S = 0***

- **Again:**

$$S_C = \frac{\alpha\beta(r+\delta)}{\delta(r+\delta)+\alpha^2\gamma}$$

- **Nash equilibrium:**

$$H_i = \beta Y_i - 0.5Y_i^2 - 0.5\gamma S^2 + \lambda_i\left(\frac{\alpha}{N}\sum_{j=1}^{N} Y_j - \delta S\right)$$

- **Necessary conditions:**

$$\beta - Y_i + \frac{\alpha}{N}\lambda_i = 0, i = 1,2,...,N$$

$$\dot{\lambda}_i(t) - r\lambda_i(t) = \gamma S(t) + \delta\lambda_i(t), i = 1,2,...,N$$

- **Steady states for** *8*$_i$: ***(r + *)8$_i$ + (S = 0***

- **Steady states for *S*:** ***∀∃ + (∀^{2}/N)8$_i$ - *S = 0***

- **Steady state Nash equilibrium:**

$$S_N = \frac{\alpha\beta(r + \delta)N}{\delta(r + \delta)N + \alpha^2\gamma} > S_C$$

With Dynamic Programming?

- **For optimal control and international coordination the same (Bellman's Principle of Optimality)**

- **Hamilton/Jacobi/Bellman equation:**

$$V_{it} - rV_i +$$

$$max[\beta Y_i - 0.5Y_i^2 - 0.5\gamma S^2 + V_{iS}(\frac{\alpha}{N}\sum_{j=1}^{N} Y_j - \delta S)] = 0$$

in value function $V_i(t, S)$

- **Problem is stationary:** $V_{it} = 0$

- **Necessary condition:**

$$\beta - Y_i(S) + \frac{\alpha}{N} V_{iS}(S) = 0$$

- **Try quadratic value function:**

$$V_i(S) = -0.5\sigma_2 S^2 - \sigma_1 S + \sigma_0, \sigma_2 > 0, \sigma_1 > 0$$

- **Necessary condition:**

$$\beta - Y_i(S) - \frac{\alpha}{N}(\sigma_2 S + \sigma_1) = 0$$

- **Pollution accumulation:**

$$\dot{S}(t) = \alpha\beta - \alpha^2 \frac{\sigma_1}{N} - (\delta + \alpha^2 \frac{\sigma_2}{N})S(t), S(0) = S_0$$

- **Steady state:**

$$S = \frac{\alpha\beta N - \alpha^2 \sigma_1}{\delta N + \alpha^2 \sigma_2}$$

- **Heavy calculations**

$$S > S_N > S_C$$

• Terminology:

feedback **Nash equilibrium (decision *Y* depends on the state *S*), as opposed to *open-loop* Nash equilibrium, derived with maximum principle (Hamiltonians)**

• Intuition:

with feedback policies, each country reacts to higher pollution stocks with lower production and pollution; therefore each country pollutes more at the margin, because some of it will be offset by the reaction of the other countries; therefore, in equilibrium, the stock of pollution is higher

• Policy relevance:

since countries can observe the stock of pollution, feedback Nash makes more sense; an analysis with open-loop Nash underestimates coordination benefits

• Reference:

Rick van der Ploeg and Aart de Zeeuw, International aspects of pollution control, *ERE* 2, 2, 117-139, 1992

Challenge 1
Quadratic value functions?

- **Shunichi Tsutsui and Kazuo Mino, Nonlinear strategies in dynamic duopolistic competition with sticky prices, *JET* 52, 136-161, 1990**

- **Result:**
feedback Nash equilibria exist, with non-quadratic value functions (and thus non-linear strategies), with a steady state that is close to the steady state under coordination

- **For International Pollution Control: Engelbert Dockner and Ngo Van Long, *JEEM* 24, 13-29, 1993**

- **Intuition: type of trigger strategy?**

- **Return to this in Lecture 2**

Challenge 2

Linear systems?

• **Standard optimal growth model:**

$$W = \int_0^{\infty} e^{-rt}[\beta C(t) - 0.5C^2(t)]dt$$

$$\dot{K}(t) = F(K(t)) - C(t), K(0) = K_0$$

• **Necessary conditions:**

$$\beta - C - \lambda = 0$$

$$\dot{\lambda}(t) = (r - F'(K(t))\lambda(t)$$

• **The steady states for *8* are either *8* = *0* (not feasible) or *r* = *F'(K)*; if the production function *F* is concave, this equation has one solution and the phase-diagram analysis is standard again**

- If the production function F is convex-concave, this equation has two solutions and the two-dimensional system in (K, *8*) has multiple steady states

- The steady state to the right is a saddle point

- The steady state to the left is not a saddle point and cannot have limit cycles (due to the positive discount rate); the trajectories “spiral out”

- The two-dimensional system has two trajectories that satisfy the necessary conditions: one approaches the saddle point, the other “eats up all the capital”

- An analysis of the value function shows that there exists a point K_S (Skiba point), such that for $K_0 > K_S$ the first trajectory results and for $K_0 < K_S$ the second

- Reference:

A.K. Skiba, Optimal growth with a convex-concave production function, *Econometrica* 46, 527-539, 1978