The Economics of Shallow Lakes

• Lakes have conflicting functions

• Resource (services): fish, drinking water, industrial water, recreation, etc.

• Waste sink: release of phosphorus by agriculture (eutrophication)

• (Shallow) lakes have been intensively studied

• Terminology:
- oligotrophic states: high economic value of services
- eutrophic states: low economic value of services

• Flips and hysteresis observed

• Shallow lakes may be metaphor for many ecological problems

Model

$$\dot{P}(\tau) = L(\tau) - sP(\tau) + r\frac{P^2(\tau)}{P^2(\tau) + m^2}, P(0) = P_0$$

P: **accumulated amount of phosphorus**

L: **input of phosphorus ("loading")**

s: **sedimentation, outflow, other sequestration**

r: **maximum rate of internal loading**

m: **anoxic level**

- **Based on Carpenter and Cottingham (1997) and Scheffer (1997)**

- **Substitute** $x = P/m$, $a = L/r$, $b = sm/r$ **and change the time scale to** $t = r\vartheta/m$:

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{x^2(t) + 1}, x(0) = x_0$$

Static analysis

- **Constant loading *a***

- **Model-equilibrium points in *(x, a)* - plane**

$$a = bx - \frac{x^2}{x^2 + 1}$$

- **If parameter $b > 0.65$: one stable model-equilibrium for all values of *a***

- **If $0.5 < b < 0.65$: one stable model-equilibrium for high and for low values of *a*, but two stable and one unstable model-equilibria for values of *a* in between: hysteresis (flips), reversible**

- **If $b < 0.5$, one stable model-equilibrium for high values of *a*, and two stable and one unstable model-equilibria for low values of *a*:**
hysteresis (flip), but irreversible

• Welfare indicator:

$$W = ln(a) - cx^2$$

• For parameter *c* high enough, maximum welfare in oligotrophic state *(c = 1)*

• If *N* communities share the lake, welfare indicators:

$$W_i = ln(a_i) - cx^2, i = 1,2,...,N$$

• If number *N* is small, one Nash equilibrium occurs in an oligotrophic state, but if number *N* is large, two Nash equilibria occur: one in an oligotrophic state and one in a eutrophic state

• (In a repeated game context, a higher number of communities yields a stronger folk theorem: Brock and de Zeeuw, *EL*, 2002)

Dynamic analysis

- **Welfare indicators:**

$$W_i = \int_0^{\infty} e^{-rt} [ln(a_i(t)) - cx^2(t)]dt, i = 1,2,...,N$$

- **Ecological system (intermediate case, *b = 0.6*):**

$$\dot{x}(t) = \sum_{i=1}^{N} a_i(t) - bx(t) + \frac{x^2(t)}{x^2(t)+1}, x(0) = x_0$$

- **Optimal management (coordination):**

$$\frac{1}{a_i} + \lambda = 0, i = 1,2,...,N$$

$$\dot{\lambda}(t) = [(r+b) - \frac{2x(t)}{(x^2(t)+1)^2}]\lambda(t) + 2Ncx(t)$$

• **Some manipulation yields differential equation in total loading *a*:**

$$\dot{a}(t) = -[(r+b) - \frac{2x(t)}{(x^2(t)+1)^2}]a(t) + 2cx(t)a^2(t)$$

• **(Open-loop) Nash equilibrium:**

$$\frac{1}{a_i} + \lambda_i = 0, i = 1,2,\ldots,N$$

$$\dot{\lambda}_i(t) = [(r+b) - \frac{2x(t)}{(x^2(t)+1)^2}]\lambda_i(t) + 2cx(t), i = \ldots$$

• **Some manipulation yields differential equation in total loading *a*:**

$$\dot{a}(t) = -[(r+b) - \frac{2x(t)}{(x^2(t)+1)^2}]a(t) + 2\frac{c}{N}x(t)a^2(t)$$

• **Everything is driven by parameter *c/N* (and *b*)**

- **Example of *potential game* (Monderer and Shapley, GEB, 1996): Nash equilibrium with *N* communities is found by solving optimal control with parameter *c/N***

- **For *c = 1*, optimal management phase diagram has one steady state, a saddle point**

- **For *c = 1* and *N = 2* (or *N > 2*), (open-loop) Nash equilibrium phase diagram has three steady states: one oligotrophic saddle point, one eutrophic saddle point and one steady state in between from which the trajectories "spiral out"**

- **Skiba point: history matters**
(compare to Krugman, History versus expectations, *QJE*, 1991)

- **Proof in reference:**
Karl-Göran Mäler, Anastasios Xepapadeas and Aart de Zeeuw, The economics of shallow lakes, *ERE* 26, 4, 603-624, 2003
(companion paper by Brock and Starrett, same issue)

Feedback Nash equilibrium?

- **Hamilton/Jacobi/Bellman equation:**

$$rV(x) = max[ln(a_i) - cx^2 +$$

$$V_x(x)(a_i + (N-1)h(x) - bx + \frac{x^2}{x^2+1})]$$

where $a_j = h(x)$ is the feedback policy

- **Necessary condition:**

$$\frac{1}{a_i} = -V_x(x), a_i := h(x) = \frac{-1}{V_x(x)}$$

- **Hamilton/Jacobi/Bellman equation becomes:**

$$rV(x) = ln(h(x)) - cx^2 - \frac{1}{h(x)}(Nh(x) - bx + \frac{x^2}{x^2+1})$$

• **Differentiation and some manipulation yields an ordinary differential equation in *h(x)***

• **If steady state x^F were known, state equation gives boundary condition for that differential equation**

$$h(x^F) = \frac{1}{N}\left(bx^F - \frac{(x^F)^2}{(x^F)^2 + 1}\right)$$

• **Degree of freedom: multiple steady states**

• **Optimal management steady state? Trajectory? Welfare?**

• **Further research**

Summary

- **Resources/pollution: stocks and common property**
- **Optimal control and differential games**
- **Assess benefits of coordination**
- **Linear-quadratic case**
- **Open-loop and feedback Nash equilibrium**
- **International pollution control**
- **Non-convexities: convex – concave constraint**
- **Skiba point**
- **Shallow lake**