Robust Control in Water Management

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Since surface water flows are often stochastic, there is a role to be played by groundwater or surface reservoirs in protecting users against uncertainty. In most of the literature, the word "uncertainty" means the realization of an event whose true probability distribution is known. Pure uncertainty, where the state space of outcomes is known but one is unable to assign probabilities, has largely been ignored. When the decision maker is unsure about his model, in the sense that there is a range of approximate models that he also considers as possibly true, the problem is one of robust dynamic control.
The purpose of our work is to analyse water storage and use decisions in a robust framework. Uncertainty about the behaviour of precipitation is introduced, and the implications for water use are presented. Robust choices are compared with those of a benchmark stochastic model and the emergence of precautionary behaviour is discussed.

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1 Introduction

In most water systems there are multiple sources of water, with different availabilities and quality levels, so that a typical case will combine whatever surface water supplies are available (rainfall, stream flows, surface water reservoirs) between them and also with groundwater resources. Economic models of conjunctive use consider at least two sources of water, one of which is a flow and one a stock. For instance, the literature that analyses management of groundwater stocks normally includes conjunctive use (see the review on the topic by Provencher (1995)). Taking into account that surface water flows are often stochastic, there is a role to be played by groundwater or surface reservoirs in protecting users against uncertainty. Tsur (1990) studies the buffer role of groundwater in a static setting and shows that it is positive under standard concavity assumptions of the benefit function, and Tsur and Graham-Tomasi (1991) provide a similar analysis for a dynamic setting. Knapp and Olson (1995) also consider surface water variability, as well as Provencher and Burt (1994), which identify the risk externality associated with common property situations. Roseta-Palma (2000) extends the analysis by incorporating water quality as a relevant parameter. There is a paper on irreversibility (Tsur and Zemel (1997)) where the size of stock below which groundwater use becomes unfeasible is unknown, but conjunctive use is not considered. Two other papers that consider uncertainty but not conjunctive use are Fisher and Rubio (1997) and Rubio and Castro (1996).

In all the literature referred so far, the word uncertainty is taken to describe the realization of an event for which the true probability distribution is known. Thus the expected utility framework can be used. However, this type of problem was traditionally considered one of risk. Pure uncertainty, where the state space of outcomes is known but the decision maker is unable to assign probabilities, has largely been ignored in recent economic literature. Nonetheless, as stressed in Woodward and Bishop (1997), in many cases pure uncertainty might be closer to the truth, for instance when a panel of experts is consulted, since a group of people with divergent beliefs will normally not be able to reach a consensus on probability distributions. Their paper analyses circumstances under which rational choices are based in the most extreme possible outcomes, rather than on midpoint values. It also discusses the intermediate case, where some information on the set of probability distributions is known. Gilboa and Schmeidler (1989) show that introducing an axiom of uncertainty aversion as a property of the preference relation is equivalent to solving a maxmin model under a set of possible probability measures, where the decision maker maximizes over the choice variables for the worst possible case. The construction of the appropriate set of measures, however, is not discussed.

Along similar lines, a dynamic approach for problems of choice under uncertainty is presented in Hansen and Sargent (2002) (discrete time setting), Hansen, Sargent, Turmuhambetova and Williams (2002) (continuous time setting). The idea is that the decision maker is unsure about his model, in the sense that there is a group of approximate models that he also considers as possibly true. These are obtained by disturbing a benchmark model, and the admissible disturbances will reflect the set of possible probability measures that the decision maker is willing to consider. The resulting problem is one of robust dynamic control, where the objective is to choose a rule that will work under a range of different model specifications. This methodology provides a tractable way to incorporate uncertainty aversion.

The types of solutions obtained by models such as these fit quite well with the precautionary principle, which has emerged in international law as a conceptual guideline for environmental policy.[1] In fact, Woodward and Bishop (1997) mention it explicitly, whereas Hansen and Sargent (2001*a*) state that "a preference for robustness induces context-specific precaution". Considering the levels of uncertainty usually associated with climatic variables, it seems natural to exploit the instruments provided by robust control in the analysis of water storage and use decisions.

The purpose of our work is to analyse such decisions in a robust control framework of model misspecification doubts. Uncertainty about the occurrence of precipitation is introduced, and the implications for quantitative water use are presented, in both a static and a dynamic setting. Robust choices are compared with those of a benchmark stochastic model, using a linear quadratic set up as a relevant representation of a system that comprises a surface storage water reservoir. Finally, the emergence of precautionary behavior is discussed.

2 A one period model of water management

Assume that there is a user of water, who maximizes his profit by choosing the amount of water he wants. Precipitation (P) is exogenous, but there is an available source of surface water (for example, a river), from which he can take as much water as he wants at a cost, given by a cost function $c(s)$, increasing and convex. There is a water revenue function which depends on

[1] See Gollier, Julien and Treich (2000) for a discussion of the precautionary principle in an economy with a stock pollutant where there is learning about damages. The authors consider risk but not uncertainty aversion.

total water use and is increasing and concave. Maximum profit, Π, is

$$\Pi = \max_s y(w) - c(s) \quad (1)$$

where $w = P + s$

It will be assumed that when the decision on s is taken, the value for precipitation is still uncertain. In a typical stochastic problem, the agent would take P to be a random variable with

$$P = \overline{P} + \varepsilon \quad (2)$$

where ε has mean zero and variance $\sigma_\varepsilon^2 = 1$. In this very simple setting, the user maximizes the expected value of Π, and the first order condition will be $\mathcal{E} y_w = c_s$.[2] This is the usual condition that at the optimal choice the expected marginal benefit of an additional unit of surface water is equal to its marginal cost.

However, it is possible that the agent views (2) as an approximation, in the sense that he is unsure about the process that governs the behavior of precipitation.[3] One way to represent the uncertainty is to assume that he believes the process may be

$$P = \overline{P} + \varepsilon + h \quad (3)$$

where h is an unknown distortion to the mean of the shock, representing a possible specification error.

The only thing that is assumed about this distortion is that the magnitude of the square of the specification error is bounded:[4]

$$h^2 \leq \eta^2 \quad (4)$$

Given this setup, the agent wants his decision to be robust, in the sense that it will work well over a larger set of models, because he has doubts about the accuracy of the simpler approximation offered by equation 2. He wants to choose a value for s that will give him a reasonable outcome even for the worst possible value of h. Robust control provides a straightforward way to find optimal decisions that is relatively simple to solve. The uncertain problem can be thought of as a zero-sum game between two players, where

[2] In order to study interior solutions, it is assumed that the last received unit of precipitation is still revenue increasing. Thus, floods and similar situations are ruled out.

[3] The derivations in this section are similar to Hansen and Sargent (2002, chp.5).

[4] The size of the distortion must be bounded, as the agent has some information on the process. For a better explanation of this bound and its relation to the degree of uncertainty, see section 3

the agent is maximizing over s and nature is minimizing over h. Then the problem can be written as

$$\max_s \min_h \mathcal{E}(y(s+\overline{P}+\varepsilon+h)) - c(s) + \theta h^2 \tag{5}$$

where $\theta > 0$ is a fixed penalty parameter, which can be interpreted as a Lagrangian multiplier on constraint (4). First order conditions for s and h are:

$$\mathcal{E}y_w = c_s \tag{6}$$
$$\mathcal{E}y_w = 2\theta h \tag{7}$$

2.1 Quadratic case

In order to understand how a preference for robustness influences optimal choices, this section presents the case where the objective function is quadratic. Assuming quadratic revenues and linear costs, $y = a + bw - cw^2 - es$, and the approximating problem yields

$$s^* = \frac{b - 2c\overline{P} - e}{2c} \tag{8}$$

If the agent is unsure about the model in the way described above, so that equation (2) is replaced with equation (3), then the corresponding problem yields:

$$s^{**} = \frac{b - 2c\overline{P} - e + \frac{ce}{\theta}}{2c}$$
$$h^{**} = -\frac{e}{2\theta}$$

A few points can be made about the properties of this solution:

- h is negative, i.e. the worst case distortion is a smaller mean for rainfall, as expected
- when $\theta \to \infty$, $h \to 0$ and $s^{**} \to s^*$, i.e. as the agent becomes more sure about his model the solution tends to (8)[5]
- $s^{**} > s^*$, i.e. in a one period model, if the agent is unsure about his model then his response will be to pump more. Precaution in this case implies excessive pumping, in the sense that the chosen level of

[5] Notice that the model breaks down if uncertainty aversion is infinitely large ($\theta \to 0$).

surface water is optimal for the worst case h and larger than would be optimal for all other values of h. This result is compatible with the usual observation that farmers, for instance, prefer to overirrigate if water is generally available (even though it is costly) in climates with great rainfall variability.

3 Dynamic water management under model misspecification

We now turn to the dynamic problem of managing surface water when the water manager is concerned about the robustness of his/her decisions to misspecification of the model, and there is accumulation of surface water in a reservoir.

Let S_t denote the stock of surface water and let $\{B_t : t \geq 0\}$ denote a standard Brownian motion on an underlying probability space $(\Omega, \mathcal{F}, G)$. The water manager seeks to determine the optimal use of surface water. The manager's model can be stated as:

$$\max_{\{s(t)\}} \mathcal{E}_0 \int_0^{\infty} e^{-\delta t} \left[u\left(s_t + P_t \right) \right] dt \tag{9}$$

subject to

$$dS_t = (\alpha P_t - s_t - qS_t)dt \tag{10}$$

$$dP_t = \sigma dB_t \tag{11}$$

where $u\left(s_t + P_t\right) = y\left(s_t + P_t\right) - c(s)$, δ is the discount rate, α is percentage of precipitation that ends up as stream flow, q denotes losses from the surface water reservoir and σ reflects precipitation variability.

Following Hansen and Sargent (2002), Hansen et al. (2002), (9)-(11) is regarded as a benchmark model. If it was assumed that the water manager was sure about the benchmark model then there would be no concerns about robustness to model misspecification. Otherwise, concerns for robustness to model misspecification can be reflected by a family of stochastic perturbations to the Brownian motion, so that the probabilities implied by (11) are distorted. The measure G is replaced by another probability measure Q. The main idea is that stochastic processes under Q will be difficult to distinguish from those under G using a finite amount of data. The perturbed model is constructed by replacing B_t in (11) with

$$B_t = \hat{B}_t + \int_0^t h_s ds \tag{12}$$

where $\left\{\hat{B}_t : t \geq 0\right\}$ is a Brownian motion and $\{h_t : t \geq 0\}$ is a measurable drift distortion. Thus, changes in the distribution of B_t will be parametrized as drift distortions to a fixed Brownian motion $\left\{\hat{B}_t : t \geq 0\right\}$. The distortions will be zero under the measure G, in which case B_t and $\hat{B}_t$ coincide.

Therefore the water manager's concerns about misspecification of the model describing the evolution of precipitation can be expressed in the distorted model

$$dP_t = \sigma h_t dt + \sigma d\hat{B}_t \tag{13}$$

As shown in Hansen et al. (2002) the discrepancy between the distributions G and Q is measured as the relative entropy

$$R(Q) = \int_0^\infty e^{-\delta u} \mathcal{E}_Q \left(\frac{|h_u|^2}{2} \right) du \tag{14}$$

If $R(Q)$ is finite then

$$Q \left\{ \int_0^t |h_u|^2 \, du < \infty \right\} = 1$$

and Q is locally absolutely continuous with respect to G. Local absolute continuity means that for the water manager it is difficult to distinguish between the probability distributions G and Q associated with precipitation, even though the two probability distributions could be distinguished with infinite data.

Under model misspecification, equation (11) is replaced by (13). Two robust control problems can be associated with the problem of maximizing (9) subject to (10) and (13), a multiplier robust control problem and a constraint robust control problem. The multiplier robust problem in this case is defined as

$$J(\theta) = \sup_s \inf_h \mathcal{E} \int_0^\infty e^{-\delta t} u(s_t + P_t) \, dt + \theta R(Q) \tag{15}$$
$$\text{subject to (10) and (13)}$$

whereas the more intuitive constraint robust problem is defined as

$$J(\eta) = \sup_s \inf_h \mathcal{E} \int_0^\infty e^{-\delta t} u(s_t + P_t) \, dt \tag{16}$$
$$\text{subject to (10), (13) and } R(Q) \leq \eta$$

In both models the process $\{h_t : t \geq 0\}$ belongs to a set H such that the implied Q has finite entropy or $R(Q) < \infty$. In the constraint model η is the maximum specification error that the water manager is willing to accept, while in the multiplier model the robustness parameter can be interpreted as the Lagrangian multiplier associated with constraint $R(Q) \leq \eta$. The robustness parameter takes non negative values, $\theta \geq 0$, and will be zero if the constraint is inactive or infinity if the constraint is violated. A value $\theta = +\infty$ signifies no preference for robustness, while lower values for θ indicate such a preference.

Combining (14) and (15) the multiplier robust control model can be written as

$$\sup_s \inf_h \mathcal{E} \int_0^\infty e^{-\delta t} \left[u(s_t + P_t) + \theta \frac{h^2}{2} \right] dt \tag{17}$$

subject to (10) and (13)

As in section 2, in problem (17) the water manager is the maximizing agent that chooses surface water s_t to maximize utility, while nature is the minimizing agent that chooses the worst case distortion to precipitation. Using Fleming and Souganidis (1989) on the existence of a recursive solution to the multiplier problem, Hansen et al. (2002) show that problem (17) can be transformed into a stochastic infinite horizon two-player game where the Bellman-Isaacs conditions imply that the value function $W(S, P, \theta)$ satisfies

$$\delta W(S, P; \theta) = \max_s \min_h \left\{ \begin{array}{c} \left[u(s_t + P_t) + \theta \frac{h^2}{2} \right] + W_S(\alpha P_t - s_t - qS_t) \\ + W_P \sigma h + \frac{1}{2}\sigma^2 W_{PP} \end{array} \right\} \tag{18}$$

A solution for game (18) for any given value of the robustness parameter θ will determine the optimal "robust" surface water management policy. Moreover, this policy will coincide with the solution for problem 16 and it will not depend on the chosen timing protocol. Thus, the solution is time consistent, in the sense that the decision maker will stick to the original optimal plan whatever the actual state of the world.[6]

3.1 Quadratic case

Adopting a quadratic benefit function for net profits from water use, $u(s_t + P_t)$ is specified as

$$u(s_t + P_t) = a + b(s_t + P_t) - c(s_t + P_t)^2 - es_t \tag{19}$$

[6]For more on time consistency, see Hansen and Sargent (2001*b*).

The first order conditions that determine the optimal feedback rules for s and h are:

$$s_t^* = \frac{b - 2cP_t - e - W_S\left(S_t, P_t; \theta\right)}{2c} \tag{20}$$

$$h_t^* = -\frac{\sigma W_P\left(S_t, P_t; \theta\right)}{\theta} \tag{21}$$

>From (21) it is clear that if $\theta \to \infty$ then $h_t^* \to 0$ indicating that there is no preference for robustness and the manager acts as if he knows the model with certainty.

Substituting (20) and (21) into (18) we obtain the partial differential equation for the value function. Because of the linear quadratic structure of the problem we restrict attention to the class of quadratic value functions, or

$$W\left(S, P\right) = \gamma_0 + \gamma_1 S + \gamma_2 S^2 + \gamma_3 P + \gamma_4 P^2 + \gamma_5 SP \tag{22}$$

Substituting (22), (20) and (21) into (18) we obtain the parameters of the value function, which can be solved for numerically.

A possible set of parameter values is:[7]

$$\begin{aligned} &\alpha = 0.4, b = 6.316347, c = -0.000348993, e = 0.1, \\ &\sigma = 0.5, \delta = 0.05, a = 16342.668, q = 0.1 \end{aligned}$$

Considering these values, the implied feedback rules for the optimal allocation of surface water are:

$s_t^*\left(S_t, P_t \left|\theta = 50\right.\right) = 335.08 - 0.24273 P_t + 0.59486 S_t$
$s_t^*\left(S_t, P_t \left|\theta = 100\right.\right) = 431.85 - 0.4116 P_t + 2.8788 S_t$
$s_t^*\left(S_t, P_t \left|\theta \to \infty\right.\right) = 431.93 - 0.41161 P_t + 2.8788 S_t$

As expected, surface water always decreases with precipitation and increases with the amount of stored water. Furthermore, the more robust decisions imply lower water use in general, as shown in the figures 1 and 2. Note that as P_t decreases, total water use $(s_t + P_t)$ decreases as well, more so for higher levels of uncertainty.

Thus, in a dynamic context where the accumulation of water is taken into consideration, a search for robustness induces water savings. When the water manager feels that there is a large degree of uncertainty about the behavior of precipitation, he decides to use less water as a precaution, since in a dynamic setting the worst case scenario would be to excessively deplete the stored water reserves.

[7] The numerical solution assumes a finite time scenario.

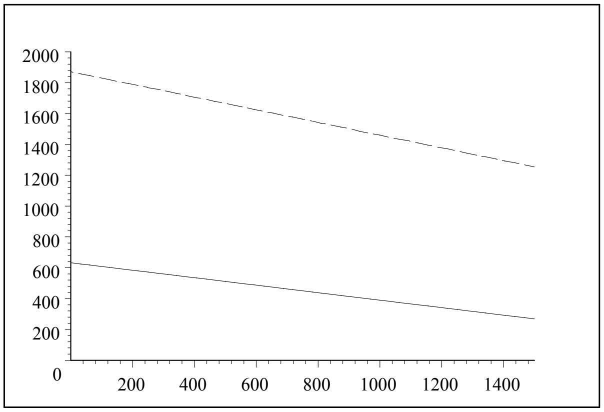

Figure 1: $s^*(P)$ for $S = 500$, with model uncertainty ($\theta = 50$, solid) and without model uncertainty ($\theta \to \infty$, dashed)

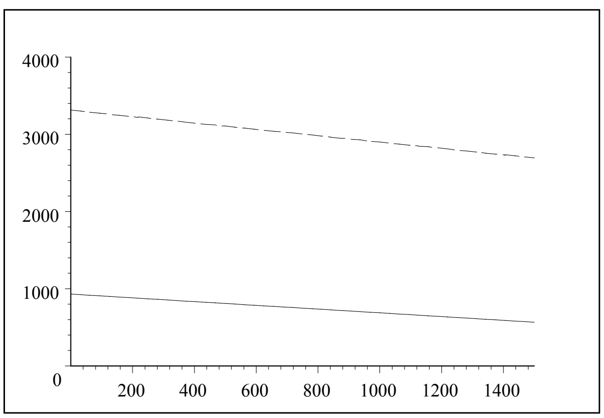

Figure 2: $s^*(P)$ for $S = 1000$, with model uncertainty ($\theta = 50$, solid) and without model uncertainty ($\theta \to \infty$, dashed)

4 Conclusion

In this paper we explore the implications of uncertainty aversion for water management, using robust control theory. The assumption that precipitation follows a stochastic process which isn't perfectly known to the decision maker seems a much better fit to what is observed in reality than the usual model of risk where the stochastic process for rainfall is known and agents take it as given in expected utility maximization.

Previous authors have pointed to the emergence of precautionary behavior when robust models are used. In the one period case, precaution implies excessive use of surface water, as there are no future penalties for the use of such a source. When a dynamic setting is considered, the water manager incorporates the possible depletion of water reserves under worst case rainfall shortcomings, and precautionary behavior then implies lower surface water applications.

In this paper, results for water use were presented for varying levels of model uncertainty, as expressed by levels of the penalty parameter θ, to explore and illustrate the implications of this particular type of methodology. However, in general θ should be chosen iteratively considering some acceptable detection error probability for distinguishing between the approximating model and the worst case model. More specifically, given each θ, the associated detection error probability can be calculated for a given sample using the likelihood ratios when the approximating model generates the data and when the worst case model is true (see Hansen and Sargent (2002, chp. 13)). Nonetheless, the purpose of this paper was to identify the possibilities of the robust control methodology and its connection to uncertainty aversion and the precautionary principle. Further empirical research such be dedicated to estimating detection error probabilities considering actual historical values values of stochastic processes such as $\{P_t\}$.

Finally, throughout the paper, the case of excessive surface water was ruled out as utility was assumed to be increasing in water use. Further research should examine the possibility of occurrence of all types of extreme events, perhaps including thresholds for irreversible situations. The application of the model to cases of high seasonal variability would also be useful.

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